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A TREATISE ON ENGINE BALANCE USING EXPONENTIALS

BY

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PREFACE.

I VENTURE to hope that the study of Engine-Balance, as presented in this work, will be found of interest to engineers. The introduction of exponentials not only permits of the ordinary engine-balance topics being discussed with remarkable ease, but it enables one to examine with little labour the more complex types of aviation engines. In the design of crank-systems with specified balance, the exponential method gives, very elegantly, results of great importance.

The replacement of the connecting-rod by concentrated masses at the crank-pin and wrist-pin, together with an infinitesimal mass at an infinite distance away, clarifies the whole analysis of the effect of the rod.

The equal and opposite couples on the crankshaft and frame due to explosion are expressed in the form of a Fourier Series, so that it becomes possible to combine at once this torque with the torques due to inertia. In engines with gear-boxes fitted to the engine-frame, the reaction-torque on the gear-box can be readily calculated and compounded with the inertia- and explosion-torques as already determined, when a complete statement of frame torque is desired.

Attention may be called to the harmonic analysis of the motion of machine pieces by manipulation of the trigonometric expressions through exponentials. The relations established among the coefficients in the harmonic expressions can be used with advantage in their computation. The ordinary engine piston-displacement coefficients are given in forms which converge very rapidly.

I gratefully here record my thanks to the Editor of "Engineering" for permission to make use of a paper on the Balance of Reciprocating Masses which I contributed to that journal.

I gladly take the opportunity of expressing my sincere thanks to Professor W. McF. Orr, D.Sc., F.R.S., to whom I am under the greatest technical indebtedness for his help in the reading of the proofs.

P. CORMAC.

College of Science for Ireland,
March, 1923.

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CHAPTER I.

THE COMPLEX NUMBER.

§ 1. **The Parallelogram Law.**—It is known that if two straight lines be drawn from a point representing in size, direction, and sense, forces acting on a particle, and a parallelogram be constructed having these two lines as adjacent sides, the diagonal drawn from the point mentioned will represent the resultant completely. If, instead of forces, the two sides of the parallelogram represent quantities of a magnitude which has size, as well as direction in space, the diagonal is defined as the vector sum of the two quantities. A vector is defined as any magnitude which has size as well as direction in space. Examples of vectors are the displacement of a particle, velocity, acceleration, momentum, force, strain.

Applying the parallelogram law to the composition of two forces acting in the same direction of, say, a_0 and $2a_0$ units, we find the resultant to be a force of $3a_0$ units. This may be expressed symbolically thus:

$$a_0 + 2a_0 = 3a_0 \quad . \quad . \quad . \quad . \quad (1)$$

in which it is understood that the + sign indicates the geometric or vector sum, that is, the + sign represents the operation of combining the quantities a_0 and $2a_0$ by the parallelogram law. Comparing the vector equation (1) with the algebraic equation

$$a_0 + 2a_0 = 3a_0$$

it is evident that when two forces act in the same direction, their vector sum may be obtained by application of the laws of common algebra.

Compounding two forces which are equal and opposite, by the parallelogram law, we find the resultant to be zero. In this case the vector equation becomes

$$a_0 + a_{180} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

in which the subscripts indicate the angles (in degrees) made by the forces with some standard direction, datum line or initial line, here for convenience taken as coinciding with the direction of one of the forces. On comparing (2) with the algebraic equation

$$a_0 + a_0 \times -1 = 0$$

we find that if we write $a_{180} = a_0 \times -1$, the equation (2) can be summed by the rules of elementary algebra. To have $a_0 \times -1 = a_{180}$, we must regard multiplication of a vector by -1 as an operation which rotates the direction of the vector through 180°. With this interpretation of -1, the fundamental laws of algebra for expressions containing -1 may be shown to hold. For example, from this definition of -1, we have

$$(a_0 \times -1) \times -1 = a_{180} \times -1 = a_0$$

This is consistent with the algebraic equation

$$a_0 \times -1 \times -1 = a_0$$

§ 2. **Interpretation of $\sqrt{-1}$.**—Taking $\sqrt{-1}$ to be defined by the equation

$$a \times \sqrt{-1} \times \sqrt{-1} = a \times -1$$

multiplication by $\sqrt{-1}$ may be expected to represent an operation which, if repeated, amounts to a rotation of the direction of the operand through 180°, that is, multiplication of a vector by $\sqrt{-1}$ rotates the direction of the vector through 90°. This interpretation can be shown to provide for the application of the fundamental laws of algebra to expressions containing $\sqrt{-1}$. As is usual, we shall denote $\sqrt{-1}$ for shortness by i.

§ 3. **Cyclic Factors.**—Since the force $OZ = a_{120}$, which makes 120° with OX, Fig. 1 may be set equal to the vector sum of a force $\frac{1}{2}a$, making 180° with OX and a force $\frac{1}{2}\sqrt{3}a$ making 90° with OX, we can write the vector equation

$$a_{120} = \tfrac{1}{2}a_{180} + \tfrac{1}{2}\sqrt{3}a_{90}$$

$$\therefore\ a_{120} = \tfrac{1}{2}a_0 \times -1 + \tfrac{1}{2}\sqrt{3}a_0 \times \sqrt{-1}$$

$$= a_0(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i).$$

Denoting $(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i)$ for shortness by ω, actual multiplication shows that $\omega^3 = 1$. This result corresponds to the fact that three successive rotations through 120° make a rotation through 360°. Since $a_{120} = a_0\omega$, ω is a factor which gives a rotation through 120°. We also have $a_0\omega^2 = a_{240}$; $a_0\omega^3 = a_{360} = a_0$. Thus the series of powers of ω : ω, ω^2, ω^3 give rotations through 120, 240, 360 degrees respectively. The series of powers of i : i, i^2, i^3, i^4, give rotations through 90, 180, 270, 360 degrees respectively. Factors having this property are called cyclic factors. It may be here noted that the cyclic factor -1 is one (the primitive) root of the equation $x^2 = 1$; the cyclic factor i which gives a rotation through one quarter revolution is a root of the equation $x^4 = 1$; the cyclic factor ω which gives a rotation of one-third of a revolution is a root of $x^3 = 1$. Generally, the cyclic factor which gives a rotation $1/n$ of a revolution is the primitive root of the equation $x^n - 1 = 0$.

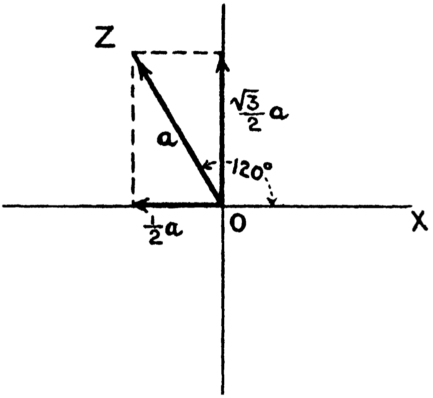

FIG. 1.—Vector analysis.

§ 4. **Cyclic Factor for Rotation through Angle θ.**—If a force a make an angle θ with our standard direction OX (Fig. 2), we can resolve it along and at right angles to the standard direction. The component along the reference axis is $a \cos \theta$, and the component at right angles is $a \sin \theta$, so that we have

$$\begin{aligned} a_\theta &= a_0 \cos \theta + a_{90} \sin \theta \\ &= a_0 \cos \theta + a_0 i \sin \theta \\ &= a_0(\cos \theta + i \sin \theta). \end{aligned}$$

Hence the cyclic factor which rotates a_0 through an angle θ is $\cos \theta + i \sin \theta$. Thus if $\theta = 120°$, we have the cyclic factor

$\omega = \cos\ 120^\circ + i\ \sin\ 120^\circ = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ as before. The above form of the cyclic factor is referred to as the polar form.

§ 5. **Exponential Form of Cyclic Factor.**—Let

$$z = \cos\theta + i\sin\theta.$$

Then
$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = iz.$$

If $e^{i\theta}$ be defined as in Art. 8, Example 12, the solution of the equation $\frac{dz}{d\theta} = iz$ is $z = e^{i\theta} + C$, and since $z = 1$ when $\theta = 0$, we must have $C = 0$. Hence we get $z = e^{i\theta}$. We therefore have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

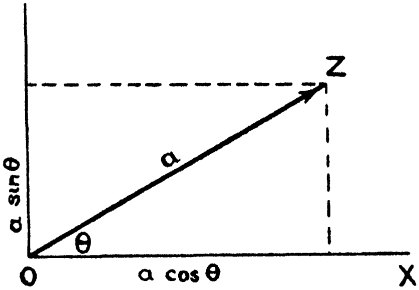

FIG. 2.—The vector $a(\cos\theta + i\sin\theta)$.

Changing θ into $-\theta$, we get $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$, or

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

From these equations, by addition and subtraction, we get

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})\ ;\ \sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Also, by putting $\theta = \pi$ we get $e^{i\pi} = -1$; $e^{-i\pi} = -1$; $e^{2i\pi} = 1$; $e^{\pm 2ki\pi} = 1$.

§ 6. **Multiplication and Division by a Cyclic Factor.**—A number of the form $r(\cos\phi + i\sin\phi)$, or its equivalent $re^{i\phi}$, is called a complex number. The absolute value of a complex number $re^{i\phi}$ is r ; the absolute value is often called the modulus ; the angle ϕ is called the angle or argument. Expressed in the form of a complex number, the force OZ (Fig. 1) is $ae^{\frac{2}{3}i\pi}$; the force OZ (Fig. 2) is $ae^{i\theta}$. In each case the absolute value or modulus is a; in the first case the argument is 120°, in the second example it is θ. Writing

$$re^{i\phi} \times e^{i\theta} = re^{i(\phi+\theta)}$$

shows that multiplication by $e^{i\theta}$ rotates the quantity $re^{i\phi}$ anti-clockwise through an angle θ. As

$$re^{i\phi} \div e^{i\theta} = re^{i(\phi-\theta)}$$

we see that division by $e^{i\theta}$ rotates the operand in the negative (clockwise) sense through an angle θ.

§ 7. **Graphical Representation.**—When the complex number is in the form $re^{i\theta}$, or $r(\cos\theta + i\sin\theta)$, it can be plotted by drawing and marking off r units along a line making an angle θ with the initial line. Thus in Fig. 2, $a_\theta \equiv ae^{i\theta}$ is plotted by drawing from the pole O, a line OZ, a units long, and making an angle θ with the initial line OX. By writing $r\cos\theta = u$, $r\sin\theta = v$, the complex number $r(\cos\theta + i\sin\theta)$ becomes $u + iv$. To plot the complex number when given in this form, we assume a set of rectangular co-ordinate axes in the xy-plane (Fig. 3), having the x-axis coinciding with the initial line: the line OQ from the origin to the point $x = u$, $y = v$, then represents $u + iv$. The absolute value or modulus of $u + iv$, denoted by $|u + iv|$ is the magnitude of OQ, that is, $\sqrt{(u^2 + v^2)}$. The angle or argument is $\tan^{-1} v/u$.

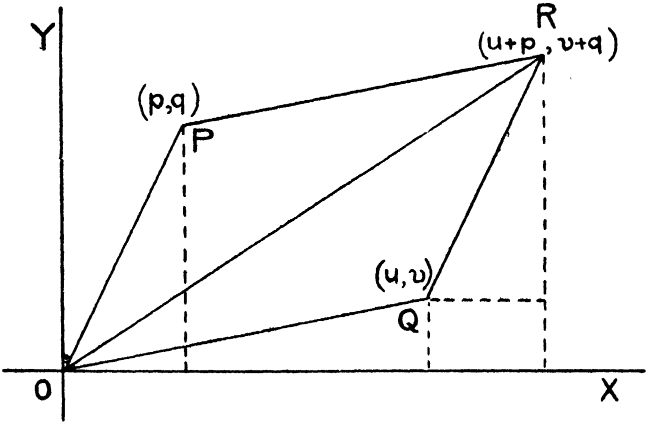

FIG. 3.—Addition of complex numbers.

The sum of two complex numbers $u + iv$, and $p + iq$, is

$$(u + iv) + (p + iq) = (u + p) + (v + q)i$$

These numbers are shown plotted in Fig. 3; OQ represents $u + iv$; OP represents $p + iq$; and OR represents $(u + p) + (v + q)i$. The figure shows that the line OR is the diagonal of the parallelogram of which OQ and OP are sides. Hence by using complex numbers to represent forces, we can obtain their resultant by adding the complex numbers in accordance with the laws of real algebra.

§ 8. **Illustrative Examples.**

1. Obtain the cyclic factor for a rotation of 120°.
2. Obtain the cyclic factor for a rotation through 72°. Write down the roots of the equation $x^5 - 1 = 0$.
3. Denoting $\frac{1}{2}(\sqrt{-3} - 1)$ by ω show that $a_0(1 + \omega + \omega^2) = 0$; that is, the resultant of three equal forces along cranks at 120° is zero.
4. Show that $a_0(1 + \omega^2 + \omega^4) = 0$, i.e., the resultant of equal forces along cranks which have rotated through twice the angles of the cranks in Ex. 3 is zero.
5. Show that $a_0(1 + \omega^n + \omega^{2n}) = 3a_0$ or 0, according as n is or is not a multiple of 3.
6. Show that $a_0(1 + i)$ is a vector $1{\cdot}414a_0$, making an angle of 45° with a_0.
7. Show that

$$\sin mx \cos nx = \tfrac{1}{2} \sin (m + n)x + \tfrac{1}{2} \sin (m - n)x$$
$$-\sin mx \sin nx = \tfrac{1}{2} \cos (m + n)x - \tfrac{1}{2} \cos (m - n)x$$
$$\cos mx \cos nx = \tfrac{1}{2} \cos (m + n)x + \tfrac{1}{2} \cos (m - n)x.$$

8. Show that the modulus of the product of two numbers is the product of the moduli, and that the modulus of the quotient of two numbers is the quotient of the moduli.
9. Plot and find the modulus and angle in the following cases:

$$-724 - 276i; \quad -276 - 724i;$$
$$-426 - 301i; \quad 513 + 411i.$$

10. Find by algebraic manipulation the sum

$$ae^{i\theta_1} + be^{i\theta_2} + ce^{i\theta_3} \text{ for } a = 1;\ b = 2;\ c = 3;\ \theta_1 = 0^\circ;$$
$$\theta_2 = 30^\circ;\ \theta_3 = 120^\circ. \text{ Verify result graphically}$$

11. Obtain graphically the sum Ex. 10 when $\theta_1 = 5^\circ$; $\theta_2 = 55^\circ$; $\theta_3 = 165^\circ$; and by obtaining in polar form the cyclic factors for above angles, check result algebraically.

12. From $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \ldots$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots$$

develop the results of section 5. Note that $e^{i\theta}$ is defined as the series obtained by putting $i\theta$ for x in the expansion of e^x.

13. Write in polar form the complex numbers which express the values of $e^{2i\pi/3}$, $e^{4i\pi/3}$, $e^{i\pi/4}$, e^i. Plot these numbers.

14. Express in the form $u + iv$ (rectangular form) the numbers in Ex. 13 correct to three decimal places.

15. Find graphically, with what accuracy you can, the vector sum of the series represented by e^i. Compare result with the plot obtained in Ex. 13.

CHAPTER II.

ROTATING MASSES.

§ 9. **Rotating Masses.**—In Fig. 4 OM is a crank rotating with angular velocity ω about O, and carrying a mass m at M. The displacement of m is given by a distance r = OM measured along OX, and then rotated through an angle θ. The displacement of m is thus given by the vector

$$z = re^{i\theta}.$$

The velocity of m is

$$\dot{z} = \frac{d}{dt}\left(re^{i\theta}\right) = ir\frac{d\theta}{dt}.\ e^{i\theta} = ir\omega e^{i\theta}.$$

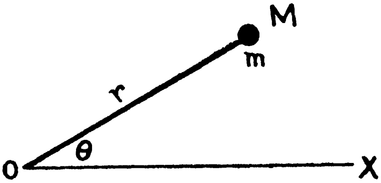

Fig. 4.—Rotating mass.

Assuming ω constant, we get the acceleration of m to be

$$\ddot{z} = \frac{d}{dt}\left(ir\omega e^{i\theta}\right) = i^2r\omega^2e^{i\theta} = -\ r\omega^2e^{i\theta}.$$

This acceleration develops in m an inertia resistance (centrifugal force) $mr\omega^2e^{i\theta}$, that is, the mass multiplied by the acceleration. The inertia resistance of a rotating mass (uniform rotation being assumed) is therefore an outward force along the crank of amount $mr\omega^2$. It is the resultant of such inertia forces we are concerned with in the analysis of the balance of rotating masses.

§ 10. **Illustrative Example 1.**—Determine the resultant inertia resistance F developed by two masses m_1, m_2, on cranks at 180°, radii r_1, r_2 respectively.

The inertia resistance due to m_1 (Fig. 4) is

$$\omega^2 m_1 r_1 e^{i\theta}.$$

The inertia resistance developed by m_2 is

$$\omega^2 m_2 r_2 e^{i(\theta+\pi)} = \omega^2 m_2 r_2 e^{i\theta} \,.\, e^{i\pi} = -\,\omega^2 m_2 r_2 e^{i\theta}.$$

The resultant F is therefore

$$F = \omega^2(m_1 r_1 e^{i\theta} - m_2 r_2 e^{i\theta}) = \omega^2(m_1 r_1 - m_2 r_2)e^{i\theta},$$

that is, a force $\omega^2(m_1 r_1 - m_2 r_2)$ along the crank to m_1. Evidently F is zero, and the masses are "in balance" if $m_1 r_1 = m_2 r_2$.

Example 2.—What mass must be placed on a crankshaft opposite the crank-pin to "balance" a mass of 920 lbs. at the crank-pin; crank radius, 13 inches, radius to balance mass 18 inches. This problem occurs in the balancing of outside cylindered locomotive engines. In locomotive work it is customary to figure as rotating at crank-pin the actual rotating mass, together with two-thirds of the reciprocating mass.

Example 3.—The centre of gravity of a fly-wheel 16 inches diameter, mass 80 lbs., is $\frac{1}{16}$ inch from the centre. Find what mass should be taken from the rim to secure its balance.

Example 4.—Show that masses m_1, m_2, m_3 on coplanar cranks, radii r_1, r_2, r_3, at 120°, are in balance if $m_1 r_1 = m_2 r_2 = m_3 r_3$.

Since, if the masses be in balance at any speed, they will be in balance at all speeds, we may take the angular velocity to be unity.

The resultant inertia resistance is then

$$\begin{aligned} & m_1 r_1\left\{e^{i\theta} + e^{i\left(\theta+\frac{2\pi}{3}\right)} + e^{i\left(\theta+\frac{4\pi}{3}\right)}\right\} \\ &= m_1 r_1 e^{i\theta}\{1 + e^{2i\pi/3} + e^{4i\pi/3}\} \\ &= m_1 r_1 e^{i\theta}(1 + \omega + \omega^2) \text{ where } \omega = \tfrac{1}{2}(\sqrt{-3} - 1) \\ &= 0. \end{aligned}$$

Example 5.—A system of discrete masses m_1, m_2, m_3, . . . rotate at radii r_1, r_2, r_3, . . . which at a given instant make angles θ_1, θ_2, θ_3, . . . with a fixed direction. Find a mass m_0 at a radius r_0 necessary to secure the balance of the rotating system.

We require the total inertia force to be zero,

$$\therefore\ \omega^2(m_1 r_1 e^{i\theta_1} + m_2 r_2 e^{i\theta_2} + m_3 r_3 e^{i\theta_3} + \ .\ .\ .\ + m_0 r_0 e^{i\theta_0}) = 0$$

$$\therefore\ m_0 r_0 e^{i\theta_0} = -\,(m_1 r_1 e^{i\theta_1} + m_2 r_2 e^{i\theta_2} + m_3 r_3 e^{i\theta_3} + \ .\ .\ .)$$

We therefore obtain the vector sum on the right and reverse its sign; the absolute value of the sum gives m_0r_0, while the argument of the member on the right gives the angle which r_0 makes with the fixed direction at the instant considered.

The vector sum may be obtained graphically, or if written in polar form, the sum may be obtained analytically. See Examples 10, 11, Article 8.

§ 11. **Masses Rotating in Different Planes.**—When the rotating masses lie in different planes, we have but to consider the shaft on which they rotate as a beam attacked by forces which represent the dynamic effects of the rotating masses. The general problem is that of balancing any mass system by masses in two given planes perpendicular to the shaft. To determine the balance masses, assume the shaft to be supported by bearings placed in the given planes. Then, considering the shaft as a beam attacked by external forces as indicated above, obtain the reactions on the journals. The balance masses, now, are such that their rotation develops inertia forces equal to these reactions.

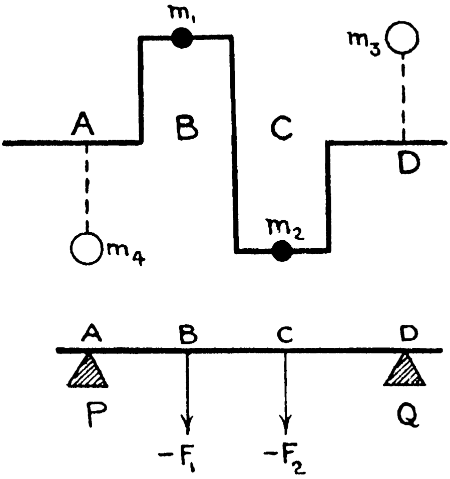

FIG. 5.—Balancing of rotating masses on cranks at 180°.

§ 12. **Illustrative Example.**—The masses m_1, m_2 on cranks at 180° are to be balanced by masses rotating in the planes through A and D (Fig. 5).

In this and in the examples immediately following, it is convenient to take the vertical as our standard direction;

also, we take the angular velocity of the shaft to be unity.

The inertia resistances developed by m_1, m_2, are:

$$F_1 = m_1 r_1 e^{i\theta_1}\ ;\ F_2 = m_2 r_2 e^{i(\theta_1+\pi)} = -m_2 r_2 e^{i\theta_1}$$

As F_1 and F_2 are referred to the vertical they are upward forces. They are equivalent to the downward forces $-F_1$, $-F_2$ shown as beam loads in the equivalent beam problem (Fig. 5). Taking moments about P, we get for the equilibrium of the beam,

$$Q\ .\ DA = -F_2\ .\ CA - F_1\ .\ BA.$$
$$\therefore Q = e^{i\theta_1}(m_2 r_2\ .\ CA - m_1 r_1\ .\ BA)/DA.$$

The cyclic factor $e^{i\theta_1}$ shows that the direction of Q is the same as that of the crank to m_1. We have now to find a mass m_3 which, rotating at a radius r_3 in the plane through D, will develop an inertia resistance Q. We therefore have

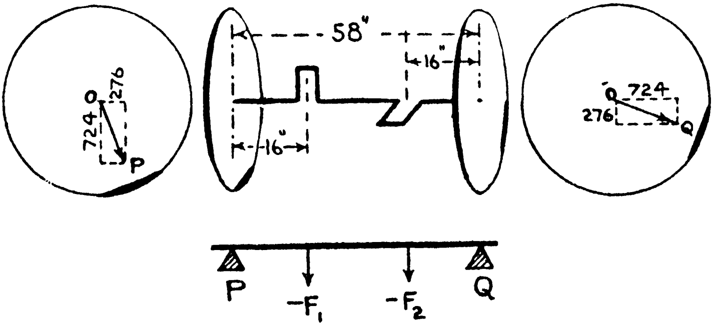

Fig. 6.—Balancing of rotating masses on cranks at 90°.

$$m_3 r_3 e^{i\theta_1} = Q = e^{i\theta_1}(m_2 r_2 CA - m_1 r_1 BA)/DA,$$

from which, if r_3 be assigned, we can determine the balance mass m_3. In like manner, we obtain m_4.

§ 13. **Balancing Locomotive.**—Fig. 6 shows the driving wheels of a locomotive 58 inches apart, left crank vertical and right crank perpendicular to plane of paper. The cylinder centre lines are 26 inches apart and 16 inches from the wheels. Assuming masses of 1000 lbs. rotating with each crank, which is 1 foot radius, determine balance masses on wheels.

Since the value of ω will not affect the result, we shall assume $\omega = 1$.

The inertia force due to left crank is $F_1 = 1000$ poundals. As the right crank makes 90° with the vertical, the inertia force due to it is $F_2 = 1000\,i$ poundals. Both F_1 and F_2 are positive when referred to the vertical direction. They are equivalent to the downward forces $-F_1$, $-F_2$, of the beam problem, corresponding to the balancing problem, shown in lower part of Fig. 6. Taking moments about Q, we get

$$\begin{aligned} 58P &= -F_1 \,.\, 42 - F_2 \,.\, 16 \\ &= -42{,}000 - 16{,}000\,i \\ \therefore P &= -724 - 276\,i. \end{aligned}$$

A vector representing P is shown drawn on the left wheel. The balance mass m_3 is placed on this vector at a radius r_3 such that $m_3 r_3 = OP = \sqrt{724^2 + 276^2} = 775$ lbs.-ft.

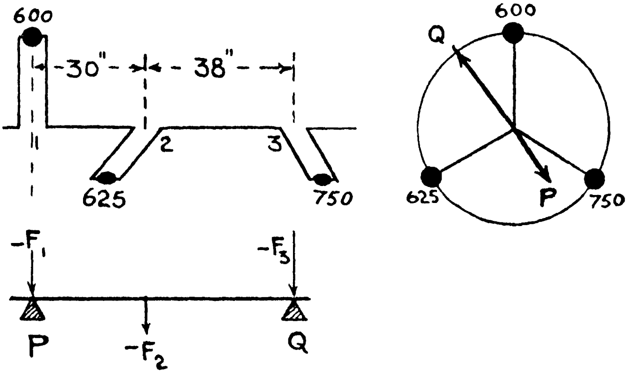

Fig. 7.—Balancing of rotating masses on cranks at 120°.

Taking moments about P we get

$$\begin{aligned} 58\,Q &= -F_2 42 - F_1 16 \\ &= -42{,}000\,i - 16{,}000 \\ \therefore Q &= -724\,i - 276. \end{aligned}$$

A vector representing Q is shown drawn on the loose wheel on the right. The balance mass m_4 is placed on this vector at a radius r_4 such that $m_4 r_4 = OQ = \sqrt{724^2 + 276^2} = 775$ lbs.-ft.

§ 14. **Balancing Cranks at 120°.**—A compound engine having three cranks at 120° has balance masses rotating in the planes of the extreme cranks, which are 68 inches apart. Taking the masses to be balanced as 600 lbs. and 750 lbs. on the outer cranks, and 625 lbs. on the inner crank, which is 38 inches from the heaviest crank, find the balance masses if all radii be 1 foot.

The inertia loads F_1, F_2, F_3 are 600, 625 ω, 750 ω^2, where $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$. Fig. 7 shows a beam equivalently loaded, the loads being $-F_1$, $-F_2$, $-F_3$ downwards.

Taking moments about Q, for the equilibrium of the beam, we get

$$\begin{aligned} 68P &= -68F_1 - 38F_2 \\ &= -68 \cdot 600 - 38 \cdot 625(-1 + \sqrt{3}i)\tfrac{1}{2} \\ \therefore\ P &= -600 + 174 - 301i = -426 - 301i. \end{aligned}$$

Taking moments about P we get

$$\begin{aligned} 68Q &= -68F_3 - 30F_2 \\ &= -68 \cdot 750(-1 - \sqrt{3}i)\tfrac{1}{2} - 30 \cdot 625(-1 + \sqrt{3}i)\tfrac{1}{2} \\ \therefore\ Q &= 375 + 650i + 138 - 239i = 513 + 411i. \end{aligned}$$

Vectors representing P and Q, plotted from the above values are shown in the end view (Fig. 7). The arms carrying the balance masses will lie along these lines. As the radius of rotation is 1 foot, the absolute values of P and Q give the balance masses, which are therefore $\sqrt{426^2 + 301^2}$, or 521 lbs., and $\sqrt{513^2 + 411^2}$, or 659 lbs., respectively.

§ 15. **General Problem of Rotating Masses.**—The general problem in which the cranks make angles θ_1, θ_2, θ_3 . . . with the vertical is solved in the same manner. We write the inertia force due to the *n*th crank as an upward force :

$$F_n = m_n(\cos\theta_n + i\sin\theta_n),$$

and consider this as a downward load,

$$-F_n = -m_n(\cos\theta_n + i\sin\theta_n)$$

on beam. The reactions of the beam supports due to all such loads give the inertia resistances of masses which, rotating in planes through the supports, secure the balance of the given system.

CHAPTER III.

SYMMETRICAL ENGINE.

A. *Inertia Forces.*

§ 16. **Piston Displacement, Velocity and Acceleration.**—In the slider crank mechanism (Fig. 8), let r be the crank radius, l the length of the connecting rod, x the displacement of the piston from the centre of the crankshaft, θ and ϕ the angles made by the crank and rod with the line of stroke. From the figure we have, by projection on the line of stroke,

$$x = r \cos \theta + l \cos \phi = r(\cos \theta + \rho^{-1} \cos \phi).$$

Also, we have $r \sin \theta = l \sin \phi$, hence $\sin^2 \phi = \rho^2 \sin^2 \theta$

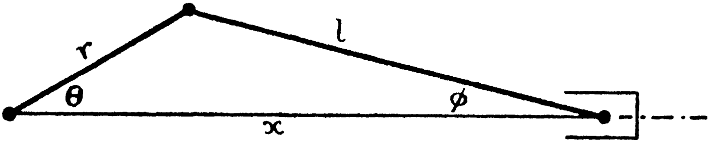

FIG. 8.—Crank and connecting rod.

where ρ is written for the ratio r/l. Since $\cos^2 \phi = 1 - \sin^2 \phi$, we have

$$\rho^{-1} \cos \phi = \rho^{-1}(1 - \rho^2 \sin^2 \theta)^{\frac{1}{2}} = q_0 + q_2 \cos 2\theta + q_4 \cos 4\theta + \ldots$$

(Appendix 15, 16).

Hence we have the piston displacement

$$x = r(\cos \theta + q_0 + q_2 \cos 2\theta + q_4 \cos 4\theta + q_6 \cos 6\theta + \ldots).$$

The piston velocity and acceleration, obtained by differentiation, assuming the angular velocity ω of the crank to be uniform, are

$$\frac{dx}{dt} = -r\omega(\sin \theta + 2q_2 \sin 2\theta + 4q_4 \sin 4\theta + \ldots)$$

$$\frac{d^2x}{dt^2} = -r\omega^2(\cos \theta + 4q_2 \cos 2\theta + 16q_4 \cos 4\theta + \ldots).$$

§ 17. **Inertia-Force.**—If m be the mass of the reciprocating parts, we have the inertia force F developed by the above acceleration, given by

$$F = -m\frac{d^2x}{dt^2} = mr\omega^2(\cos\theta + p_2\cos 2\theta + p_4\cos 4\theta + \ldots)$$

p_n being written for n^2q_n. In this series for F, the first term, i.e., $mr\omega^2\cos\theta$, is spoken of as the primary inertia-force; the second term, $p_2mr\omega^2\cos 2\theta$, is referred to as the secondary inertia-force, or second harmonic; the term $p_nmr\omega^2\cos n\theta$ gives the nth harmonic. The force F is along the line of stroke, and hence for a cylinder in line engine having the lines of stroke of the several cylinders parallel, the resultant inertia-force, being the resultant of a number of parallel forces, is the algebraic sum of the inertia forces developed in each cylinder. In like manner, the total inertia-force of the nth order is the algebraic sum of the nth harmonics of all the cylinders. In the following discussion we shall consider only those engines in which the reciprocating mass, crank radius, and connecting rod length are the same for all cylinders (or engines in which the product of reciprocating mass by crank radius, and the quotient of connecting rod by crank, are the same for all).

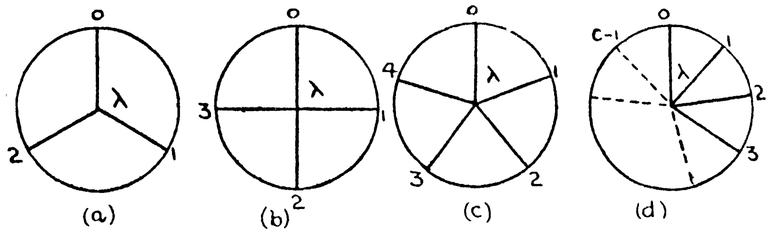

FIG. 9.—Cranks in end view.

§ 18. **Multicylinder-in-Line Engine.**—We shall here consider engines having in end view their crank-pins evenly distributed on the circumference of the crank circle. For example, in Fig. 9 (*a*) represents an engine having three cranks at 120°; (*c*) an engine having five cranks at 72°. Plainly, an engine with c cranks will have its cranks at an angle λ such that $c\lambda = 2\pi$. The cranks 1, 2, 3 . . . $c - 1$, will make angles λ, 2λ, 3λ . . . $(c-1)\lambda$, with crank No. 0. When the crank No. 0 makes an angle θ with the line of stroke, the angles made by the cranks 1, 2, 3 . . . $c - 1$ are $\theta + \lambda$; $\theta + 2\lambda$; $\theta + 3\lambda$;

. . . $\theta + (c - 1)\lambda$, respectively. In this configuration the nth harmonic inertia-force for each cylinder is

Cylinder 0	$mr\omega^2 p_n \cos n\theta$.
Cylinder 1	$mr\omega^2 p_n \cos n(\theta + \lambda)$.
Cylinder 2	$mr\omega^2 p_n \cos n(\theta + 2\lambda)$.
. . . .	. . . .
Cylinder $(c - 1)$	$mr\omega^2 \,.\, p_n \cos n(\theta + \overline{c - 1}\lambda)$.

Substituting the exponential values of the circular functions, this becomes

Cylinder 0	$\frac{1}{2}mr\omega^2 p_n\{e^{ni\theta} + e^{-ni\theta}\}$.
Cylinder 1	$\frac{1}{2}mr\omega^2 p_n\{e^{ni(\theta+\lambda)} + e^{-ni(\theta+\lambda)}\}$.
Cylinder 2	$\frac{1}{2}mr\omega^2 p_n\{e^{ni(\theta+2\lambda)} + e^{-ni(\theta+2\lambda)}\}$.
. . . .	. . . .
Cylinder $(c - 1)$	$\frac{1}{2}mr\omega^2 p_n\{e^{ni(\theta+\overline{c-1}\lambda)} + e^{-ni(\theta+\overline{c-1}\lambda)}\}$.

For the complete engine, due to the c cylinders, the total inertia force of the nth order is therefore

$$
\begin{aligned}
F_n = {} & \tfrac{1}{2}mr\omega^2 p_n\{e^{ni\theta} + e^{-ni\theta} + e^{ni(\theta+\lambda)} + e^{-ni(\theta+\lambda)} + e^{ni(\theta+2\lambda)} \\
& \quad + e^{-ni(\theta+2\lambda)} + \ldots + e^{ni(\theta+\overline{c-1}\lambda)} + e^{-ni(\theta+\overline{c-1}\lambda)}\}. \\
= {} & \tfrac{1}{2}mr\omega^2 p_n\{e^{ni\theta}(1 + e^{ni\lambda} + e^{2ni\lambda} + \ldots + e^{\overline{c-1}ni\lambda}) \\
& \quad + e^{-ni\theta}(1 + e^{-ni\lambda} + e^{-2ni\lambda} + \ldots + e^{-\overline{c-1}ni\lambda})\}. \\
= {} & \tfrac{1}{2}mr\omega^2 p_n\{e^{ni\theta}(1 + e^{ni\lambda} + e^{2ni\lambda} + \ldots + e^{\overline{c-1}ni\lambda}) \\
& \quad + e^{-ni\theta} \,.\, e^{-\overline{c-1}ni\lambda}(1 + e^{ni\lambda} + e^{2ni\lambda} + \ldots + e^{\overline{c-1}ni\lambda})\}. \\
= {} & \tfrac{1}{2}mr\omega^2 p_n(e^{ni\theta} + e^{-ni\theta} \,.\, e^{-\overline{c-1}ni\lambda})\{1 + e^{ni\lambda} + e^{2ni\lambda} + \ldots \\
& \quad + e^{\overline{c-1}nc\lambda}\}.
\end{aligned}
$$

F_n will be zero for all values of θ when the factor in the chain brackets on right side of above equation is zero, (the factor in the circumflex brackets cannot be zero for all values of θ).

Let $\quad E = 1 + e^{ni\lambda} + e^{2ni\lambda} + \ldots + e^{\overline{c-1}ni\lambda}$.

Then $\quad E \,.\, e^{ni\lambda} = e^{ni\lambda} + e^{2ni\lambda} + \ldots + e^{cni\lambda}$.

By subtraction we get

$$
\begin{aligned}
E(1 - e^{ni\lambda}) = 1 - e^{cni\lambda} &= 1 - e^{2\pi ni} \text{ since } c\lambda = 2\pi \\
&= 1 - 1 = 0.
\end{aligned}
$$

Hence E is surely zero unless $1 - e^{ni\lambda} = 0$, that is, unless $e^{ni\lambda} = 1$. But $e^{ni\lambda} = 1$ if $ni\lambda = 2k\pi i$, that is, if $n\lambda = k \,.\, 2\pi = k \,.\, c\lambda$ (since $c\lambda = 2\pi$). Hence E is surely zero except when $n = kc$, where k is an integer.

That E is not zero when $1 - e^{ni\lambda} = 0$ is at once evident on substituting $e^{ni\lambda} = 1$ in the series E. Plainly we get

$$\begin{aligned} E &= 1 + 1 + 1 + \ldots \text{ to } c \text{ terms} \\ &= c. \end{aligned}$$

In this case then we have, since $n = kc$,

$$\begin{aligned} F_n &= \tfrac{1}{2} mr\omega^2 p_n (e^{ni\theta} + e^{-ni\theta} \,.\, e^{-\overline{c-1}kci\lambda}) \{1 + 1 + 1 + \ldots\} \\ &= \tfrac{1}{2} mr\omega^2 p_n (e^{ni\theta} + e^{-ni\theta}) c \text{ since } c\lambda = 2\pi \\ &= cmr\omega^2 p_n \cos n\theta. \end{aligned}$$

Hence for values of n, which are integral multiples of the number of cranks, we have the nth harmonics unbalanced and equal to $cmr\omega^2 p_n \cos n\theta$. For values of n which are not integral multiples of the number of cranks, F_n is zero, since E is zero.

§ 19. **Summary.**—The inertia forces of those orders only which are integral multiples of the number of cranks are unbalanced. Thus the six-cylinder petrol engine having cranks at 120° (Fig. 9 (*a*) and Fig. 44) has the inertia forces of all orders balanced, except those of the 6th, 12th, 18th, etc., orders. The four-cylinder petrol engine, with cranks at 180° (Fig. 41), has the 2nd, 4th, and, in fact, all higher order inertia-forces unbalanced. The total unbalanced nth harmonic is in general:

$$F_n = cmr\omega^2 p_n \cos n\theta.$$

Referring to the series for p_2, p_4, p_6, etc. (Appendix 15), it may be noted that these coefficients become exceedingly small for values of $n > 4$. Hence the balance of the inertia forces of the six cylinder petrol engine is very satisfactory. Since p_2 is appreciable, the balance of the four cylinder petrol engine is not good if the speed be high, the magnitude of F_n varying as the square of the speed. The maximum value of F_n is $cmr\omega^2 p_n$. In the four-cylinder engine the maximum unbalanced force is therefore not less than

$$4mr\omega^2(p_2 + p_4 + p_6 + \ldots) = 4mr\omega^2 \rho. \quad \text{(See p. 144.)}$$

If $\rho = \frac{1}{4}$ this becomes $mr\omega^2$, which is equal to the primary inertia-force of one cylinder.

In the six cylinder engine the maximum unbalanced force is

$$6mr\omega^2(p_6 + p_{12} + p_{18} + \ldots).$$

For a rod four cranks long this becomes $\cdot 0004 mr\omega^2$ approximately. (See pp. 144, 145.)

B. *Inertia-Torque.*

§ 20. **Effects of Inertia-Resistance of Piston Mass.**—The torque on the crankshaft, and reactions at cylinder walls and crankshaft bearings, occasioned by the inertia-resistance of the piston (reciprocating mass), will be identical with reactions obtained by considering the mechanism at rest and the piston head attacked by an external force equal to the inertia-resistance developed by the motion. If F be this " inertia-pressure," and S the thrust along the rod, we have, considering the equilibrium of the piston G, and resolving horizontally (Fig. 10),

$$S \cos \phi = F, \text{ or } S = F \sec. \phi.$$

Resolving the rod thrust S at the crank pin, along and perpendicular to the crank, we get for the latter (the tangential) component,

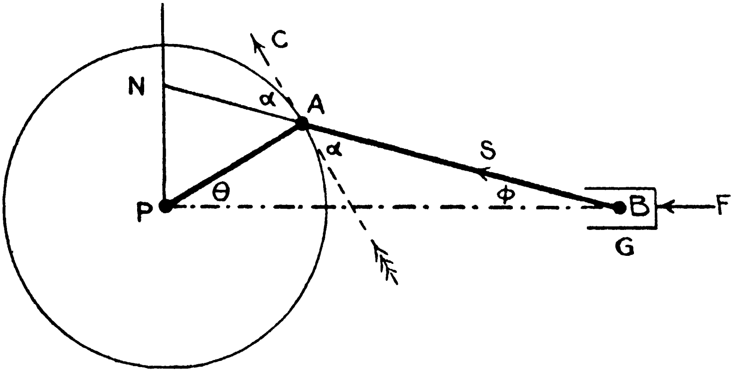

Fig. 10.—Inertia force analysis.

$$C = S \cos \alpha = F \cos \alpha / \cos \phi.$$

But $\cos \alpha = \cos NAC = \sin PAN$; $\cos \phi = \sin PNA$.

$$\therefore C = F \sin PAN / \sin PNA = F . PN/AP.$$

If T be the torque on the crankshaft, we have

$$T = C . AP = F . AP . PN/AP = F . PN.$$

This result might be obtained perhaps more directly by resolving the rod thrust S into horizontal and vertical components at N. The horizontal component, of course, is F, and this gives a torque F . PN on the crankshaft together with a horizontal force F at P. The vertical component of the rod thrust S at N is S sin ϕ. This creates a vertical pressure on the crankshaft bearings. The side thrust on the cylinder walls due to the vertical component of S at the gudgeon pin is also S sin ϕ. Hence the " inertia pressure " F makes itself evident as :—

(1) A horizontal force F on the crankshaft bearings.
(2) A torque in the crankshaft of moment F . PN.
(3) A downward load S sin ϕ on the cylinder walls, and an upward load of equal amount on the crankshaft bearings. This is, therefore, a clockwise couple on the engine frame of moment S sin ϕ . PB.

Since $S = F \sec. \phi$, this becomes

$$F \sec \phi \sin \phi \, PB = F \tan \phi \, . \, PB = F \, . \, PN,$$

that is, a couple of moment equal and opposite to the torque on the engine shaft.

Item (1) above has been considered in Part A of this chapter. We have here to consider items (2) and (3). If, as is usual, the crankshaft torque be opposed by resistances from sources or foundations other than the engine frame, then item (3) above appears as a (varying) couple tending to rock the engine frame.

§ 21. **Expression for Inertia-Torque.**—If v be the velocity of the piston G (Fig. 10), its resolved part along the rod is $v \cos \phi$. The velocity of A is (along AC) ωAP. Hence the component of A's velocity along the rod is

$$\omega AP \, . \cos a.$$

Therefore, since A and G must have the same velocity component in the direction of the rod, we get

$$v \cos \phi = \omega AP \cos a$$

$$\therefore v \sin PNA = \omega AP \, . \sin NAP.$$

$$\therefore v = \omega AP \, . \sin NAP/\sin PNA = \omega AP \, . \, NP/AP = \omega NP$$

$$\therefore v/\omega = NP.$$

Substituting in the equation T = F . PN of Art. 20, we get

$$T = Fv/\omega.$$

This is a result which could have been stated at once as following from the Principle of Work. The rate at which F does work is equal to the rate of working of T, that is, $Fv = T\omega$.

If m be the mass and f the acceleration of the piston, we have $F = mf$, so that the inertia-torque T is given by

$$T = -mfv/\omega$$

(taking the positive direction of f and v to be away from the crankshaft).

§ 22. **Series for Inertia-Torque.**—Since $\frac{d}{dt}(v^2) = 2v\frac{dv}{dt} = 2fv$, we have $fv = \frac{1}{2}\frac{d}{dt}(v^2)$. From Art. 16,

$$v = -r\omega(\sin\theta + 2q_2\sin 2\theta + 4q_4\sin 4\theta + 6q_6\sin 6\theta + \ldots)$$

$$\begin{aligned}\therefore v^2 = r^2\omega^2(&\sin^2\theta + 4q_2^2\sin^2 2\theta + 16q_4^2\sin^2 4\theta + 36q_6^2\sin^2 6\theta + \ldots\\ &+ 4q_2\sin\theta\sin 2\theta + 8q_4\sin\theta\sin 4\theta + 12q_6\sin\theta\sin 6\theta + \ldots\\ &+ 16q_2q_4\sin 2\theta\sin 4\theta + 24q_2q_6\sin 2\theta\sin 6\theta + \ldots\\ &+ 48q_4q_6\sin 4\theta\sin 6\theta + \ldots)\end{aligned}$$

To simplify this, we make use of the formula, Ex. 7, Art. 8, $\sin mx\sin nx = -\frac{1}{2}\cos(m+n)x + \frac{1}{2}\cos(m-n)x$.

$$\begin{aligned}\therefore v^2 = r^2\omega^2(&-\tfrac{1}{2}\cos 2\theta + \tfrac{1}{2} - 2q_2^2\cos 4\theta + 2q_2^2\\ &- 8q_4^2\cos 8\theta + 8q_4^2\\ &- 18q_6^2\cos 12\theta + 18q_6^2\\ &- 2q_2\cos 3\theta + 2q_2\cos\theta - 4q_4\cos 5\theta + 4q_4\cos 3\theta - 6q_6\cos 7\theta\\ &+ 6q_6\cos 5\theta - 8q_2q_4\cos 6\theta + 8q_2q_4\cos 2\theta - 12q_2q_6\cos 8\theta\\ &+ 12q_2q_4\cos 4\theta - 24q_4q_6\cos 10\theta + 24q_4q_6\cos 2\theta + \ldots)\\ = r^2\omega^2\{(&\tfrac{1}{2} + 2q_2^2 + 8q_4^2 + 18q_6^2 + \ldots)\\ &+ 2q_2\cos\theta\\ &+ \cos 2\theta(-\tfrac{1}{2} + 8q_2q_4 + 24q_4q_6 + \ldots)\\ &+ \cos 3\theta(-2q_2 + 4q_4)\\ &+ \cos 4\theta(-2q_2^2 + 12q_2q_4)\\ &+ \cos 5\theta(-4q_4 + 6q_6)\\ &+ \cos 6\theta(-8q_2q_4) + \ldots\}\end{aligned}$$

$$\therefore v^2 = r^2\omega^2(e_0 + e_1\cos\theta + e_2\cos 2\theta + e_3\cos 3\theta + e_4\cos 4\theta + \ldots)$$

Where

$$\begin{aligned}e_0 &= \tfrac{1}{2} + 2q_2^2 + 8q_4^2 + 18q_6^2 + \ldots\\ e_1 &= 2q_2\\ e_2 &= -\tfrac{1}{2} + 8q_2q_4 + 24q_4q_6 + \ldots\\ e_3 &= -2q_2 + 4q_4 -\\ e_4 &= -2q_2^2 + 12q_2q_4\\ e_5 &= -4q_4 + 6q_6\\ e_6 &= -8q_2q_4.\end{aligned}$$

Example.—Calculate e_0, e_1, e_2, e_3, e_4, e_5, e_6, for the following values of ρ:

$\rho=\frac{1}{3}$. Ans.: ·5146, ·1714, —·5004, —·1738, —·0152, ·0024, ·0004
$\rho=\frac{1}{4}$. Ans.: ·5080, ·1270, —·5001, —·1280, —·0081, ·0010, ·0001
$\rho=\frac{1}{5}$. Ans.: ·5051, ·1010, —·5001, —·1016, —·0051, ·0005, ·0001

By differentiation of the series for v^2 we obtain

$$2v\frac{dv}{dt}=2fv=-r^2\omega^3(e_1\sin\theta+2e_2\sin 2\theta+3e_3\sin 3\theta+\ldots).$$

Hence the inertia-torque

$$\begin{aligned}\mathrm{T}=-mfv/\omega&=mr^2\omega^2(\tfrac{1}{2}e_1\sin\theta+e_2\sin 2\theta+\tfrac{3}{2}e_3\sin 3\theta+\ \ldots\ )\\&=mr^2\omega^2(t_1\sin\theta+t_2\sin 2\theta+t_3\sin 3\theta+\ \ldots).\end{aligned}$$

Example.—Calculate t_1, t_2, t_3, t_4, t_5, t_6, for the following values of ρ:

$\rho=\frac{1}{3}$. Ans.: ·0857, — ·5004, — ·2607, — ·0304, ·0060, ·0012
$\rho=\frac{1}{4}$. Ans.: ·0635, — ·5001, — ·1920, — ·0162, ·0025, ·0003
$\rho=\frac{1}{5}$. Ans.: ·0505, — ·5001, — ·1524, — ·0102, ·0012, ·0003.

§ 23. **Balance of Inertia-Torque.**—Considering a c-crank engine of the type discussed in Art. 18, the inertia-torque of the nth order for the several cylinders is, when crank No. 0 makes an angle θ with the line of stroke:

Cylinder 0, $mr^2\omega^2t_n\sin n\theta$.
Cylinder 1, $mr^2\omega^2t_n\sin n(\theta+\lambda)$.
Cylinder 2, $mr^2\omega^2t_n\sin n(\theta+2\lambda)$.
. . . .
Cylinder $c-1$, $mr^2\omega^2t_n\sin n(\theta+\overline{c-1}\lambda)$.

Substituting exponential values, we get

Cylinder 0, $\frac{1}{2i}mr^2\omega^2t_n\{e^{ni\theta}-e^{-ni\theta}\}$.
Cylinder 1, $\frac{1}{2i}mr^2\omega^2t_n\{e^{ni(\theta+\lambda)}-e^{-ni(\theta+\lambda)}\}$.
Cylinder 2, $\frac{1}{2i}mr^2\omega^2t_n\{e^{ni(\theta+2\lambda)}-e^{-ni(\theta+2\lambda)}\}$.
. . . .
Cylinder $c-1$, $\frac{1}{2i}mr^2\omega^2t_n\{e^{ni(\theta+\overline{c-1}\lambda)}-e^{-ni(\theta+\overline{c-1})\lambda}\}$.

The total inertia-torque of the nth order is therefore

$$\begin{aligned}\mathrm{T}_n&=\tfrac{1}{2i}mr^2\omega^2t_n\{e^{ni\theta}-e^{-ni\theta}+e^{ni(\theta+\lambda)}-e^{-ni(\theta+\lambda)}+e^{ni(\theta+2\lambda)}\\&\qquad-e^{-ni(\theta+2\lambda)}+\ \ldots+e^{ni(\theta+\overline{c-1}\lambda)}-e^{-ni(\theta+\overline{c-1}\lambda)}\}.\\&=\tfrac{1}{2i}mr^2\omega^2t_n(e^{ni\theta}-e^{-ni\theta}\,.\,e^{-\overline{c-1}ni\lambda})\{1+e^{ni\lambda}+e^{2ni\lambda}\\&\qquad+\ldots+e^{\overline{c-1}ni\lambda}\}\end{aligned}$$

Comparing this expression with the expression for F_n, the inertia force given in Art. 18, it will be evident that T_n is zero for all values of the crank angle θ if n be not an integral multiple of c. When n is an integral multiple of c, we have for $n = kc$,

$$\begin{aligned} T_n &= \tfrac{1}{2i}\, mr^2\omega^2 t_n (e^{ni\theta} - e^{-ni\theta} \,.\, e^{-\overline{c-1}kci\lambda})\{1 + 1 + 1 + \ldots + 1\} \\ &= \tfrac{1}{2i}\, mr^2\omega^2 t_n (e^{ni\theta} - e^{-ni\theta})c \quad \text{since } c\lambda = 2\pi \\ &= c\, mr^2\omega^2 t_n \sin n\theta. \end{aligned}$$

§ 24. **Summary.**—It will be noted that both even and odd harmonics occur in the series for inertia-torque, whereas in the series for inertia-force we have but even harmonics only. The inertia-torque of those orders only which are integral multiples of the number of cranks are unbalanced. The six-cylinder petrol engine, with cranks of 120°, has the third, sixth, ninth, etc., harmonics of the inertia-torque unbalanced. It may be seen on calculation of the t_n coefficients (Art. 22), that those of the sixth, ninth, and higher orders are very small, and consequently these harmonics may be ignored.

For a six-cylinder engine having a piston 3·2 lbs., crank radius 3 inches, $\omega = 200$, corresponding to 1910 r.p.m., we have the maximum value of T_3 given by (when rod is four cranks long) :

$$\begin{aligned} T_3 &= cmr^2\omega^2 t_3 \\ &= \frac{-6 \times 3{\cdot}2 \times {\cdot}25^2 \times 200^2 \times {\cdot}1920}{32} \doteq -300 \text{ lbs.-ft.} \end{aligned}$$

Hence, due to *inertia-torque*, we have an alternating torque $+300 \sin 3\omega t$ lbs.-ft. acting on the engine frame, and an alternating torque $-300 \sin 3\omega t$ lbs.-ft. transmitted by the crankshaft.

The four-cylinder petrol engine has the second, fourth, sixth, etc., inertia-torque harmonics unbalanced. With cylinders of same size as the above six-cylinder, the maximum inertia-torque (the secondary in this case) is

$$\begin{aligned} T_2 &= cmr^2\omega^2 t_2 \\ &= \frac{-4 \times 3{\cdot}2 \times {\cdot}25^2 \times 200^2 \times {\cdot}5001}{32} \doteq -500 \text{ lbs.-ft.} \end{aligned}$$

Thus, as regards balance of inertia-torque, the four-cylinder engine compares favourably with the six-cylinder.

As a second example, we shall take $\omega = 125$ radians per sec. (1200 r.p.m.), reciprocating mass ·4 lbs. per square inch of piston head area, crank radius 2 inches. For the third harmonic of a six-cylinder we now have the maximum value of T_3:

$$T_3 = \frac{6 \times \cdot 4 \times (\frac{1}{6})^2 \times 125^2 \times \cdot 1920}{32} = 6{\cdot}25 \text{ lbs.-ft. per sq. in. piston head.}$$

For the second harmonic of a four-cylinder we have maximum value of T_2:

$$T_2 = \frac{4 \times \cdot 4 \times (\frac{1}{6})^2 \times 125^2 \times \cdot 5001}{32} = 10{\cdot}8 \text{ lbs.-ft. per sq. inch piston head.}$$

By multiplying these figures by the area of one piston we get the total inertia torque of the engine.

A single-cylinder engine has all harmonics unbalanced. Referring to Art. 22, it will be seen that the largest coefficient is the secondary. For one cylinder of the engine of the first example above, the maximum secondary inertia-torque is approximately 125 lbs.-ft.

C. *Inertia-Torque due to Rod.*

§ 25. **Equivalent Dynamical System.**—The connecting rod is dynamically equivalent to any mass system which has:

(1) The same total mass as the rod.
(2) The same position of centre of mass as rod.
(3) The same moment of inertia (about axis through the centre of mass) as rod.

We can, therefore, regard the rod of mass $m = m_1 + m_2$ as equivalent to—

(1) A mass m_1 at the crank pin and m_2 at the gudgeon pin, such that $m_1 l_1 = m_2 l_2$, where l_1 and l_2 are the distances of the crank pin and gudgeon pin from the centre of mass of the rod.

(2) A mass Δm distributed on a circle of radius r, centre G the centre of gravity of the rod; the mass Δm is assumed very small and r very big, but such that $\Delta m \,.\, r^2 = I'$, where I' is the difference between the moment of inertia of the rod (about G) and the moment of inertia $(m_1 l_1^2 + m_2 l_2^2)$ of the masses m_1 and m_2 (about G). It is assumed that the fictitious rim carrying Δm is rigidly connected with the rod. It will be observed that the conditions providing for equality of mass, moment of inertia, and position of mass centre of rod and its equivalent

system are satisfied. It is understood that the rod is symmetrical with respect to its plane of motion, and that the reduced mass system lies in the plane of symmetry; also, that the centre of mass lies on the line joining the crank and gudgeon pins. The mass m_2 may be added to the piston mass, and m_1 to the crank-pin mass, and so dealt with as reciprocating and rotating masses respectively, the effects of which have already been discussed. We have here but to consider the effect of the rim of moment of inertia I' about its centre. The rim offers no inertia-resistance to the translatory motion of the rod, since for such motion we may assume the rim's mass Δm to be concentrated at its centre of gravity, and we have stipulated that Δm is indefinitely small. The total effect of the rim is therefore that due to its angular motion.

§ 26. **Inertia-Resistance Couple.**—The angular acceleration of the rod is $\ddot{\phi}$. This will develop in the rim an inertia-resistance torque $T = -I'\ddot{\phi}$, which will occasion loads F, F at the crank and gudgeon pins (Fig. 11). The reaction at the gudgeon pin being perpendicular to the line of stroke, the loads F, F are necessarily perpendicular to the line of stroke. To determine F, we have

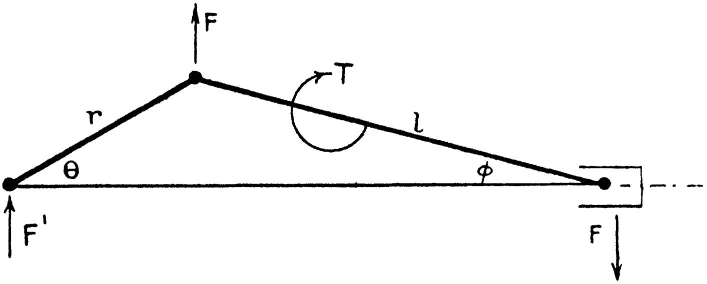

FIG. 11.—Analysis of inertia couple on rod.

$$Fl \cos \phi = T.$$

The force F on the crank-pin gives an anti-clockwise torque $F \cdot r \cos \theta$ on the crankshaft, together with an equal force F′ on the crankshaft bearing. This latter force F′, with the force F on the gudgeon pin, give a clockwise couple on the engine-frame of moment $F(r \cos \theta + l \cos \phi)$. The inertia torque T, developed by the angular acceleration of the rod, therefore appears as a couple $-Fr \cos \theta = -\dfrac{Tr \cos \theta}{l \cos \phi}$ on the crankshaft, and a couple $F(r\cos\theta + l\cos\phi) = T + \frac{Tr\cos\theta}{l\cos\phi}$ on the engine-frame. From Fig. 11 we have

$$r\sin\theta = l\sin\phi.$$
$$\therefore r\cos\theta\,.\frac{d\theta}{dt} = l\cos\phi\,.\frac{d\phi}{dt}.$$
$$\therefore \frac{r\cos\theta}{l\cos\phi} = \frac{d\phi}{d\theta} = \dot{\phi}/\omega.$$

Hence the crankshaft couple is $-T\dot{\phi}/\omega$, and the engine frame couple $T + T\phi/\omega$. Since $T = -I'\ddot{\phi}$, we get the crankshaft couple C given by

$$C = I'\ddot{\phi}\dot{\phi}/\omega,$$

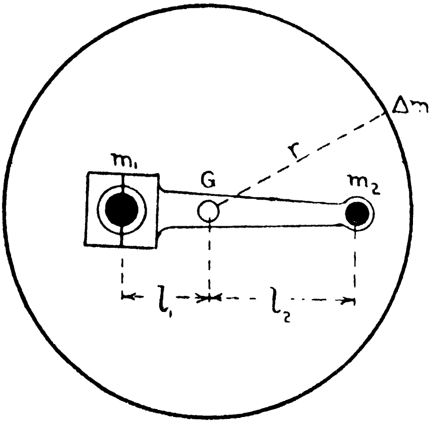

FIG. 12.—Equivalent dynamical system.

and the engine-frame couple given by

$$S = -I'\ddot{\phi} - I'\ddot{\phi}\dot{\phi}/\omega.$$

§ 27. **Balance of Rods: Multicylinder Engine.**—From Art. 26 we have

$$\phi = r\omega\cos\theta/l\cos\phi = \rho\omega\cos\theta(1 - \sin^2\phi)^{-\frac{1}{2}}$$
$$= \rho\omega\cos\theta(1 - \rho^2\sin^2\theta)^{-\frac{1}{2}}.$$

Hence, from Appendix 18, we have

$$\phi\ddot{\phi} = -\tfrac{1}{2}\omega^3(2s_2\sin 2\theta + 4s_4\sin 4\theta + \ldots).$$

$\therefore$ Crankshaft torque $C = -I'\omega^2(s_2\sin 2\theta + 2s_4\sin 4\theta + \ldots)$.

From Appendix 17 we have

$$\ddot{\phi} = -\omega^2(u_1 \sin\theta + 3u_3 \sin 3\theta + 5u_5 \sin 5\theta + \ldots).$$

$$\therefore \text{ Frame torque } S = I'\omega^2(u_1 \sin\theta + s_2 \sin 2\theta + 3u_3 \sin 3\theta + 2s_4 \sin 4\theta + \ldots).$$

Reference to Art. 23 is sufficient to show that only the harmonics which are integral multiples of the number of cranks are unbalanced in a multicylinder engine.

I′ is usually small and negative. A Hispano-Suiza rod 8·75 inches long gave mass at gudgeon pin ·83 lbs.; mass at crank-pin 2·14 lbs.; I of rod minus I of detached masses $= I' = -\cdot 09$ ft.²-lb. units. At an engine speed 230 radians per sec. (2200 r.p.m.), the primary torque on frame due to one of these rods is $-\cdot 09 \,.\, 230^2 \,.\, \cdot 2520 \sin\theta$ poundals ft., or approximately $37 \sin\theta$ lbs.-ft. The secondary torque is $-\cdot 09 \,.\, 230^2 \,.\, \cdot 0312 \sin 2\theta$, or $4\cdot 5 \sin 2\theta$ lbs.-ft. This is also the secondary torque on the crankshaft.

Exercise.—A Harris-Corliss engine, 26⅛ inches × 60 inches. R.P.M., 60. Length of rod, 150 inches; weight, 1200 lbs. Weight of other reciprocating parts, 1300 lbs. Radius of crank, 2·5 feet. Principal polar radius of gyration, 48 inches. Distance of c.g. from wrist pin, 78 inches. Distance of centre of oscillation from c.g., 29·57 inches. Determine equivalent detached masses at crank and gudgeon pins, and moment of inertia I′ of the rim of mass Δm. What are the crankshaft and frame torques due to I′?

§ 28. **Kinetic Energy of Mechanism.** The kinetic energy of the reciprocating mass is $\frac{1}{2}mv^2$. From Art. 22 we have

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\omega^2(e_0 + e_1 \cos\theta + e_2 \cos 2\theta + e_3 \cos 3\theta + \ldots).$$

The kinetic energy of the rod is most conveniently dealt with by substituting the equivalent dynamical system as in Art. 25. The mass m_1 at the crank-pin contributes $\frac{1}{2}m_1r^2\omega^2$ to the kinetic energy. The mass m_2 at the gudgeon pin may be taken into consideration by adding m_2 to the piston mass. The kinetic energy due to the rim of moment of inertia I′ is $\frac{1}{2}I'\dot{\phi}^2$. From Appendix 18 we have

$$\tfrac{1}{2}I'\dot{\phi}^2 = \tfrac{1}{2}I'\omega^2(s_0 + s_2 \cos 2\theta + s_4 \cos 4\theta + \ldots).$$

The total kinetic energy of one cylinder may therefore be put in the form

$$W = M\omega^2(E_0 + E_1 \cos\theta + E_2 \cos 2\theta + E_3 \cos 3\theta + \ldots).$$

For a multicylinder engine the harmonics of W, which are not multiples of the number of cranks, cancel out. Thus in the four-cylinder petrol engine, with cranks at 180°, the odd harmonics cancel out; in the six-cylinder, with cranks at 120°, only harmonics which are multiples of three occur.

D. *Explosion-Torque.*

§ 29. **Indicator Diagram.**—We here develop a formula giving the crankshaft torque occasioned by the fluid pressure in the cylinder. It will be assumed that this pressure follows the law

$$PV^{1\cdot 3} = K,$$

where in the working stroke

P = pressure of fluid in lbs. per square inch;
V = 25 + percentage of stroke completed;
K = 28870.

On the compression stroke K is taken = 7978. It will be observed that the compression ratio taken is 125/25 = 5. The assumed value of K will be found to give a mean explosion pressure of 110 lbs. per square inch. Table I., page 35, gives the volume swept out by piston, and the piston velocity/crank-pin velocity corresponding to various crank angles. The figures were calculated from the appropriate series established in the Appendix. Table II. gives the volume of fluid in cylinder for various crank angles, this volume being obtained by adding the clearance volume (25) to the volume swept out by piston given in Table I. The explosion and compression pressures corresponding to the volumes given in Table II. were then calculated from the p-v formulæ given above. Table III. sets out the pressures obtained in Table II. all diminished by 15 lbs. per square inch (atmospheric pressure). This gives the resultant fluid pressure on the piston. The crank-effort occasioned by the pressures were then calculated from the relation, Art. 21:

$$T\omega = Pv,$$

where T is the crank torque, and v the piston velocity. As the crank effort $E = T/r$, we have $E = Pv/r\omega$. In the tables E is the crank effort for an engine with pistons 1 square inch area, or 1·128 inch diameter.

§ 30. **Pairing of Cylinders.**—The examination of the explosion torque of the four-stroke-cycle multicylinder engine can be simplified in many cases by regarding the complete engine as consisting of a number of twin-cylinder units, such that for each unit we get one explosion per revolution of the crankshaft. Thus, the usual four-cylinder petrol engine will consist of two such units—cylinders 2 and 3 giving one unit, and cylinders 1 and 4 the other unit. The six-cylinder petrol engine will consist of three such units, viz., cylinders 1, 6; 2, 5; 3, 4.

§ 31. **Series for Crank-Effort.**—The crank-effort E per twin will go through the twelve values set out in Table III., page 37, during each revolution of the crank through 2π. A curve passing through these values of E, when plotted against θ, is given by

$$E = a_0 + a_1 \cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + a_4\cos 4\theta + a_5\cos 5\theta + a_6\cos 6\theta$$
$$+ b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + b_4\sin 4\theta + b_5\sin 5\theta,$$

and may be taken to represent the actual crank effort. The coefficients are determined by the solution of twelve simultaneous equations.

§ 32. **Determination of Coefficients.**—Denoting by y_n the crank effort for the crank angle $n\pi/12$, we have from the equation of the previous Art.:

$$y_0 = a_0 + a_1\cos 0 + a_2\cos 0 + \ldots + a_6\cos 0 + b_1\sin 0 + \ldots + b_5\sin 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$y_1 = a_0 + a_1\cos 30 + a_2\cos 60 + \ldots + a_6\cos 6\times 30 + b_1\sin 30 + \ldots + b_5\sin 5\times 30 \quad . \quad . \quad . \quad (2)$$
$$y_2 = a_0 + a_1\cos 60 + a_2\cos 120 + \ldots + a_6\cos 6\times 60 + b_1\sin 60 + \ldots + b_5\sin 5\times 60 \quad . \quad . \quad . \quad (3)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$y_{11} = a_0 + a_1\cos 11\times 30 + a_2\cos 2\times 11\times 30 + \ldots + a_6\cos 6\times 11\times 30 + b_1\sin 11\times 30 + \ldots + b_5\sin 5\times 11\times 30 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

By addition we get

$$\begin{aligned} y_0 + y_1 + y_2 + \ldots + y_{11} = 12a_0 &+ a_1(\cos 0 + \cos 30 + \ldots + \cos 11\times 30) \\ &+ a_2(\cos 0 + \cos 60 + \ldots + \cos 11\times 60) \\ &+ a_3(\cos 0 + \cos 90 + \ldots + \cos 11\times 90) \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &+ a_6(\cos 0 + \cos 180 + \ldots + \cos 11\times 180) \\ &+ b_1(\sin 0 + \sin 30 + \ldots + \sin 11\times 30) \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &+ b_5(\sin 0 + \sin 150 + \ldots + \sin 11\times 150). \end{aligned}$$

Any one of the bracketed cosine sums on the right can be written

$$\tfrac{1}{2}(1 + e^{ni\theta} + e^{2ni\theta} + \ldots + e^{11ni\theta}) + \tfrac{1}{2}(1 + e^{-ni\theta} + e^{-2ni\theta} + \ldots + e^{-11ni\theta})$$
$$= \frac{1}{2}\left(\frac{1 - e^{12ni\theta}}{1 - e^{ni\theta}}\right) + \frac{1}{2}\left(\frac{1 - e^{-12ni\theta}}{1 - e^{-ni\theta}}\right)$$
$$= 0 + 0 \text{ (since } 12\theta = 2\pi \text{ and } n\theta \neq 2\pi).$$

Similarly, any of the bracketed sine sums can be written

$$\tfrac{1}{2i}(1 + e^{ni\theta} + e^{2ni\theta} + \ldots + e^{11ni\theta}) - \tfrac{1}{2i}(1 + e^{-ni\theta} + \ldots + e^{-11ni\theta})$$
$$= \frac{1}{2i}\left(\frac{1 - e^{12ni\theta}}{1 - e^{ni\theta}}\right) - \frac{1}{2i}\left(\frac{1 - e^{-12ni\theta}}{1 - e^{-ni\theta}}\right)$$
$$= 0 - 0.$$

Hence we get

$$y_0 + y_1 + y_2 + y_3 + \ldots + y_{11} = 12a_0,$$

since all the other terms vanish.

To determine a_1, multiply each of the equations (1) to (12) by the coefficient of a_1 in that equation. We get

$$y_0 = a_0 + a_1 \cos 0 + a_2 \cos 0 + \ldots + b_1 \sin 0 + \ldots + b_5 \sin 0$$
$$y_1 \cos 30 = a_0 \cos 30 + a_1 \cos^2 30 + a_2 \cos 60 \cos 30 + \ldots + b_1 \sin 30 \cos 30 + \ldots + b_5 \sin 150 \cos 30$$
$$y_2 \cos 60 = a_0 \cos 60 + a_1 \cos^2 60 + a_2 \cos 120 \cos 60 + \ldots + b_1 \sin 60 \cos 60 + \ldots + b_5 \sin 300 \cos 60$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$y_{11} \cos 330 = a_0 \cos 330 + a_1 \cos^2 330 + a_2 \cos 660 \cos 330 + \ldots + b_1 \sin 330 \cos 330 + \ldots + b_5 \sin 1650 \cos 330.$$

Adding, we have

$$\begin{aligned} y_0 + y_1 \cos 30 + y_2 \cos 60 + \ldots + y_{11} \cos 330 &= a_0(\cos 0 + \cos 30 + \cos 60 + \ldots + \cos 330) \\ &+ a_1(\cos^2 0 + \cos^2 30 + \ldots + \cos^2 330) \\ &+ a_2(\cos 0 + \cos 60 \cos 30 + \ldots + \cos 660 \cos 330) \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ b_1(\sin 0 + \sin 30 \cos 30 + \ldots + \sin 330 \cos 330) \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ b_5(\sin 0 + \sin 150 \cos 30 + \ldots + \sin 1650 \cos 330). \end{aligned}$$

The coefficient of a_n (for values of n not equal to one) is

$$\cos 0 + \cos\theta\cos n\theta + \cos 2\theta\cos 2n\theta + \ldots + \cos 11\theta\cos 11n\theta$$

$$= \cos 0 + \tfrac{1}{2}\{\cos(n+1)\theta + \cos(n-1)\theta + \cos(2n+2)\theta + \cos(2n-2)\theta + \ldots + \cos(11n+11)\theta + \cos(11n-11)\theta\}$$

$$= \tfrac{1}{2}\{\cos 0 + \cos(n+1)\theta + \cos 2(n+1)\theta + \ldots + \cos 11(n+1)\theta\}$$

$$+ \tfrac{1}{2}\{\cos 0 + \cos(n-1)\theta + \cos 2(n-1)\theta + \ldots + \cos 11(n-1)\theta\}$$

$$= \tfrac{1}{4}\{1 + e^{(n+1)i\theta} + e^{2(n+1)i\theta} + \ldots e^{11(n+1)i\theta} + 1 + e^{-(n+1)i\theta} + e^{-2(n+1)i\theta} + \ldots + e^{-11(n+1)i\theta} + 1 + c^{(n-1)i\theta} + e^{2(n-1)i\theta} + \ldots + e^{11(n-1)i\theta} + 1 + e^{-(n-1)i\theta} + \ldots + e^{-11(n-1)i\theta}\}$$

$$= \frac{1}{4}\left\{\frac{1 - e^{12(n+1)i\theta}}{1 - e^{(n+1)i\theta}} + \frac{1 - e^{-12(n+1)i\theta}}{1 - e^{-(n+1)i\theta}} + \frac{1 - e^{12(n-1)i\theta}}{1 - e^{(n-1)i\theta}} + \frac{1 - e^{-12(n-1)i\theta}}{1 - e^{-(n-1)i\theta}}\right\}$$

$$= \tfrac{1}{4}\{0 + 0 + 0 + 0\} = 0$$

(since $12\theta = 2\pi$, and $(n \pm 1)\theta \neq 2\pi$ for the greatest values of n is 6).

The coefficient of b_n is

$$\sin 0 + \sin\theta\cos n\theta + \sin 2\theta\cos 2n\theta + \ldots + \sin 11\theta\cos 11n\theta$$

$$= \tfrac{1}{2}\{\sin(n+1)\theta - \sin(n-1)\theta + \sin 2(n+1)\theta - \sin 2(n-1)\theta + \ldots + \sin 11(n+1)\theta - \sin 11(n-1)\theta\}$$

which is readily shown $= 0$.

The coefficient of a_1, however, is

$$\tfrac{1}{2}\{(1 + \cos 0) + (1 + \cos 2\theta) + (1 + \cos 4\theta) + \ldots (1 + \cos 22\theta)$$
$$= \tfrac{1}{2}(12 + \cos 0 + \cos 2\theta + \cos 4\theta + \ldots + \cos 22\theta)$$
$$= \tfrac{1}{2}(12 + 0) = 6.$$

Hence we have

$$y_0 + y_1\cos 30 + y_2\cos 60 + \ldots + y_{11}\cos 330 = 6a_1.$$

In like manner, by multiplying each of the equations (1) to (12) by the coefficient of a_2 in that equation, and adding the resulting equations, we get

$$
\begin{aligned}
y_0 + y_1 \cos 60 + y_2 \cos 120 + \ldots + y_{11} \cos 660 \\
= a_0(\cos 0 + \cos 60 + \cos 120 + \ldots + \cos 660) \\
+ a_1(\cos 0 + \cos 30 \cos 60 + \ldots + \cos 330 \cos 660) \\
+ a_2(\cos^2 0 + \cos^2 60 + \ldots + \cos^2 660) \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+ b_1(\sin 0 + \sin 30 \cos 60 + \ldots + \sin 330 \cos 660) \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+ b_5(\sin 0 + \sin 150 \cos 60 + \ldots + \sin 1650 \cos 660).
\end{aligned}
$$

$= 6a_2$ since as before it may be shown that all the other terms vanish. By the same method we determine

$$
\begin{aligned}
6a_3 &= y_0 + y_1 \cos 90 + y_2 \cos 180 + \ldots + y_{11} \cos 990 \\
6a_4 &= y_0 + y_1 \cos 120 + y_2 \cos 240 + \ldots + y_{11} \cos 1320 \\
6a_5 &= y_0 + y_1 \cos 150 + y_2 \cos 300 + \ldots + y_{11} \cos 1650
\end{aligned}
$$

In determining the coefficient a_6 we get

$$
\begin{aligned}
y_0 + y_1 \cos 180 + y_2 \cos 360 + \ldots + y_{11} \cos 11 \times 180 \\
= a_6(\cos^2 0 + \cos^2 180 + \cos^2 360 + \ldots) \\
+ \text{terms which total zero} \\
= 12a_6.
\end{aligned}
$$

$$\therefore\ 12a_6 = y_0 + y_1 \cos 180 + y_2 \cos 360 + \ldots + y_{11} \cos 11 \times 180.$$

To determine b_1, multiply each of the equations (1) to (12) by the coefficient of b_1 in that equation, and add the twelve resulting equations. We thus get

$$
\begin{aligned}
y_0 + y_1 \sin 30 + y_2 \sin 60 + \ldots + y_{11} \sin 330 \\
= a_0(\sin 0 + \sin 30 + \sin 60 + \ldots + \sin 330) \\
+ a_1(\sin 0 \cos 0 + \sin 30 \cos 30 + \ldots + \sin 330 \cos 330) \\
+ a_2(\sin 0 \cos 0 + \sin 30 \cos 60 + \ldots + \sin 330 \cos 660) \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+ b_1(\sin^2 0 + \sin^2 30 + \sin^2 60 + \ldots + \sin^2 330) \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+ b_5(\sin 0 + \sin 30 \sin 150 + \ldots + \sin 330 \sin 1650).
\end{aligned}
$$

Here the coefficient of a_n is

$$
\begin{aligned}
&\sin 0 + \sin \theta \cos n\theta + \sin 2\theta \cos 2n\theta + \ldots + \sin 11\theta \cos 11n\theta \\
&= \tfrac{1}{2}\{\sin(n+1)\theta - \sin(n-1)\theta + \sin 2(n+1)\theta - \sin 2(n-1)\theta + \ldots \\
&\qquad \ldots + \sin 11(n+1)\theta - \sin 11(n-1)\theta\} \\
&= 0.
\end{aligned}
$$

The coefficient of b_n $(n \neq 1)$ is

$$
\begin{aligned}
&\sin 0 + \sin \theta \sin n\theta + \sin 2\theta \sin 2r\theta + \ldots + \sin 11\theta \sin 11n\theta \\
&= \tfrac{1}{2}\{\cos(n-1)\theta - \cos(n+1)\theta + \cos 2(n-1)\theta - \cos 2(n+1)\theta \\
&\qquad + \ldots + \cos 11(n-1)\theta - \cos 11(n+1)\theta\} \\
&= 0.
\end{aligned}
$$

The coefficient of b_1, however, is

$$\tfrac{1}{2}\{1 - \cos 60 + 1 - \cos 120 + \ldots + 1 - \cos 660\}$$
$$= 6.$$

Hence

$$6b_1 = y_o + y_1 \sin 30 + y_2 \sin 60 + \ldots + y_{11} \sin 330.$$

In like manner we get

$$\begin{aligned}
6b_2 &= y_o + y_1 \sin 60 + y_2 \sin 120 + \ldots + y_{11} \sin 660 \\
6b_3 &= y_o + y_1 \sin 90 + y_2 \sin 180 + \ldots + y_{11} \sin 990 \\
6b_4 &= y_o + y_1 \sin 120 + y_2 \sin 240 + \ldots + y_{11} \sin 1320 \\
6b_5 &= y_o + y_1 \sin 150 + y_2 \sin 300 + \ldots + y_{11} \sin 1650.
\end{aligned}$$

Generally, if instead of taking 12 equidistant ordinates $y_o, y_1, \ldots y_{11}$, we were to take n equidistant ordinates $y_o, y_1, \ldots y_{n-1}$ on the effort-angle curve between o and 2π, n being even, we find that

a_o is the average value of the n ordinates;
$a_{n/2}$ is the average value of the n ordinates taken alternately plus and minus;
a_k (or b_k) is twice the average value of the products formed by multiplying each ordinate by the cosine (or sine) of k times the corresponding value of θ.

§ 33. **Computing Forms.**—The labour in calculating the 12 coefficients is considerably reduced by using the computing form devised by Prof. Runge, and given below. It may be readily verified on remembering that cos 330 = cos 30, sin 330 = – sin 30, etc.

Ordinates		y_o	y_1	y_2	y_3	y_4	y_5	y_6
			y_{11}	y_{10}	y_9	y_8	y_7	
Sum		v_o	v_1	v_2	v_3	v_4	v_5	v_6
Diff.			w_1	w_2	w_3	w_4	w_5	
	v_o	v_1	v_2	v_3		w_1	w_2	w_3
	v_6	v_5	v_4			w_5	w_4	
Sum	p_o	p_1	p_2	p_3		r_1	r_2	r_3
Diff.	q_o	q_1	q_2			s_1	s_2	
	p_o	p_1				r_1	q_o	
	p_2	p_3				r_3	q_2	
Sum	l_o	l_1			Diff.	t_1	t_2	

Multipliers of the quantities in the same horizontal rows before these are entered.	Cosine Terms.				Sine Terms.		
Sin 30° = ·5 Sin 60° = ·866 Sin 90° = 1	q_2 q_1 q_0	$-p_2$ p_1 p_0 $-p_3$	t_2	l_0 l_1	r_1 r_2 r_3	s_1 s_2	t_1
Sum of 1st col. Sum of 2nd col.	... ...	... ...	... ...	... ...	... ...	... ...	... ...
Sum . . . Difference . .	$6a_1$ $6a_5$	$6a_2$ $6a_4$	$6a_3$	$12\,a_0$ $12\,a_6$	$6\,b_1$ $6\,b_5$	$6b_2$ $6b_4$	$6b_3$

Checks:

$$y_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

$$y_1 - y_{11} = (b_1 + b_5) + \sqrt{3}(b_2 + b_4) + 2b_3.$$

Result:

$$y = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta + a_5 \cos 5\theta + a_6 \cos 6\theta + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + b_5 \sin 5\theta.$$

As an example we give the calculation of the coefficients in the series for crank-effort (Art. 31). The values of y are taken from Table III., page 37, for connecting rod four cranks long.

Ordinates	0	175·50	140·30	79·85	40·42	16·60	0
		− 41·86	− 28·18	− 11·26	− 2·95	− 0·33	
Sum	0	133·64	112·12	68·59	37·47	16·27	0
Diff.		217·36	168·48	91·11	43·37	16·93	

	0	133·64	112·12	68·6	217·4	168·5	91·1
	0	16·3	37·5		16·9	43·4	
Sum	0	149·9	149·5	68·6	234·3	211·9	91·1
Diff.	0	117·3	74·5		200·5	125·1	

	0	149·9		234·3	0
	149·5	68·6		91·1	74·5
Sum	149·5	218·5	Diff.	143·2	− 74·5

Multipliers.	Cosine Terms.				Sine Terms.		
·5	37·2	−74·7 74·9			117·6		
·866	101·7				183·7	173·8 108·4	
1	0	0 −68·6	−74·5	149·5 218·5	91·1		143·2
Sum 1st col.	37·2	−74·7	−74·5	149·5	208·7	173·8	143·2
Sum 2nd col.	101·7	6·3		218·5	183·7	108·4	
Sum . .	138·9	−68·4	−74·5	368	392·4	282·2	143·2
Diff. . .	−64·5	−81		−69	25	65·4	
	a_1 23·1	a_2 −11·4	a_3 −12·4	a_0 30·7	b_1 65·4	b_2 47·0	b_3 23·8
	a_5 −10·7	a_4 13·5		a_6 −5·7	b_5 4·2	b_4 10·9	

§ 34. **Twenty-four Ordinate Scheme.**—By obtaining the crank effort for the twenty-four angles 0°, 15°, 30°, . . . 345°, we can determine the twenty-four coefficients in the equation

$$E = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_{12} \cos 12\theta$$
$$+ b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_{11} \sin 11\theta.$$

The twenty-four ordinate computing form corresponding to that on page 32 for twelve ordinates is here given :

Ordinates	y_0	y_1	y_2	. . .	y_{11}	y_{12}
		y_{23}	y_{22}	. . .	y_{13}	
Sum	v_0	v_1	v_2	. . .	v_{11}	v_{12}
Diff.		w_1	w_2	. . .	w_{11}	

	v_0	v_1	. . .	v_5	v_6	w_1	w_2	. . .	w_5	w_6
	v_{12}	v_{11}	. . .	v_7		w_{11}	w_{10}	. . .	w_7	
Sum	p_0	p_1	. . .	p_5	p_6	r_1	r_2	. . .	r_5	r_6
Diff.	q_0	q_1	. . .	q_5		s_1	s_2	. . .	s_5	

	p_0	p_1	p_2	p_3	s_1	s_2	s_3
	p_6	p_5	p_4		s_5	s_4	
Sum	l_0	l_1	l_2	l_3	k_1	k_2	k_3
Diff.	m_0	m_1	m_2		n_1	n_2	

	l_0	l_1
	l_2	l_3
Sum	g_0	g_1

$q_0 - q_4 = t_0$ $\qquad r_1 + r_3 - r_5 = u_1$

$q_1 - q_3 - q_5 = t_1$ $\qquad r_2 - r_6 = u_2$

Multipliers	Cosine Terms.								Sine Terms.					
Sin 30 = ·5			m_2		$-l_2$	l_1			k_1					
Sin 60 = ·866				m_1						k_2	n_1	n_2		
Sin 90 = 1	g_0	g_1	m_0		l_0	$-l_3$	m_0	m_2	k_3				k_1	k_3
Sum of 1st col.	...		...		...		...		...		...		...	
Sum of 2nd col.	...		...		...		...		..		...		...	
Sum . .	$24a_0$		$12a_2$		$12a_4$				$12b_2$		$12b_4$			
Diff. . .	$24a_{12}$		$12a_{10}$		$12a_8$		$12a_6$		$12b_{10}$		$12b_8$		$12b_6$	

Multipliers.	Cosine Terms.						Sine Terms.					
Sin 15° = ·259		q_5				q_1	r_1				r_5	
Sin 30 = ·5	q_4				q_4			r_2				r_2
Sin 45 = ·707		q_3		t_1		$-q_3$	r_3		u_1		$-r_3$	
Sin 60 = ·866	q_2				$-q_2$			r_4				$-r_4$
Sin 75 = ·966		q_1				q_5	r_5				r_1	
Sin 90 = 1	q_0		t_0		q_0			r_6		u_2		r_6
Sum of 1st col.	...		...		...		...		...		...	
Sum of 2nd col.	...		...		...		...		...		...	
Sum . . .	$12a_1$		$12a_3$		$12a_5$		$12b_1$		$12b_3$		$12b_5$	
Diff. . . .	$12a_{11}$		$12a_9$		$12a_7$		$12b_{11}$		$12b_9$		$12b_7$	

Checks: $y_0 = a_0 + a_1 + a_2 + a_3 + \ldots + a_{12}$

$$\tfrac{1}{2}(y_1 - y_{23}) = \cdot 259(b_1 + b_{11}) + \tfrac{1}{2}(b_2 + b_{10}) + \cdot 707\,(b_3 + b_9) + \cdot 866(b_4 + b_8) + \cdot 966(b_5 + b_7) + b_6.$$

TABLE I.

Crank Angle. Degrees.		Rod 3 Cranks long.		Rod 4 Cranks long.		Rod 5 Cranks long.	
		Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\omega r}$.	Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\omega r}$.	Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\omega r}$.
0	360	0	0	0	0	0	0
30	330	8·78	·644	8·26	·608	7·95	·586
60	300	31·25	1·010	29·68	·974	28·75	·952
90	270	58·33	1·000	56·25	1·000	55·00	1·000
120	240	81·25	2 2	79·68	·758	78·75	·780
150	210	95·38	·356	94·86	·392	94·55	·413
180	—	100	0	100	0	100	0

TABLE II.

Crank Angle.	Rod 3 Cranks long		Rod 4 Cranks long.		Rod 5 Cranks long.	
	Volume.	Pressure.	Volume.	Pressure.	Volume.	Pressure.
0	25·0	—	25	—	25	—
30	33·78	297·4	33·26	303·7	32·95	307·2
60	56·25	153·2	54·68	159·0	53·75	162·6
90	83·33	91·92	81·25	94·85	80·00	96·92
120	106·25	67·07	104·68	68·33	103·75	69·16
150	120·38	56·66	119·86	57·35	119·55	57·51
180	125·0	54·27	125·0	54·27	125·0	54·27
210	120·38	15·75	119·86	15·85	119·55	15·9
240	106·25	18·54	104·68	18·89	103·75	19·13
270	83·33	25·40	81·25	26·26	80	26·79
300	56·25	42·35	54·68	43·94	53·75	44·93
330	33·78	82·19	33·26	83·85	32·95	84·89
360	25·0	121·5	25	121·5	25	121·5

§ 35. **Crank-Effort : Multicylinder Engine.**—Considering an engine composed of c " twin " cylinder units, we have the crank-effort due to one pair given by

$$\begin{aligned} E = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_6 \cos 6\theta \\ + b_1 \sin\theta + \ldots + b_5 \sin 5\theta, \end{aligned}$$

when the crank makes an angle θ with the line of stroke. The coefficients are given in Table IV. The nth harmonic of the crank-effort, when crank No. 0 makes an angle θ with the line of stroke, and the remaining cranks make angles $\theta + \lambda$, $\theta + 2\lambda$, . . . etc., with the line of stroke, is

Cylinder 0, $a_n \cos n\theta + b_n \sin n\theta$.
Cylinder 1, $a_n \cos n(\theta + \lambda) + b_n \sin n(\theta + \lambda)$.
Cylinder 2, $a_n \cos n(\theta + 2\lambda) + b_n \sin n(\theta + 2\lambda)$.

. . . .

Cylinder $c - 1$, $a_n \cos n(\theta + \overline{c - 1}\lambda) + b_n \sin n(\theta + \overline{c - 1}\lambda)$.

The resultant crank-effort nth harmonic is the sum of the above cosine and sine terms. We have already seen in Articles 18, 23, that this sum is zero except when n is a multiple of c, in which case the sum of the cosine terms is $c \,.\, a_n \cos n\theta$, and the sum of the sine terms $c \,.\, b_n \sin n\theta$. Thus, in the four-cylinder petrol engine, for which $c = 2$, we have balance of all the odd harmonics, but none of the even harmonics is in balance. The total unbalanced secondary is

$$\begin{aligned} E_2 &= 2(a_2 \cos 2\theta + b_2 \sin 2\theta) \\ &= 2A_2 \sin (2\theta + \epsilon) \text{ where } A_2 = \sqrt{a_2^2 + b_2^2}. \end{aligned}$$

TABLE III.

Crank Angle θ.	Rod 3 Cranks long			Rod 4 Cranks long.			Rod 5 Cranks long.			Reference to Col. 4, 7, 10.
	Gauge Pressure.	Piston Vel. / ωr.	Crank Effort E.	Gauge Pressure.	Piston Vel. / ωr.	Crank Effort E.	Gauge Pressure.	Piston Vel. / ωr.	Crank Effort E.	
0	106·5	0	0	106·5	0	0	106·5	0	0	y_0
30	282·4	·644	181·8	288·7	·608	175·5	292·2	·586	171·1	y_1
60	138·2	1·010	139·6	144·0	·974	140·3	147·6	·952	140·5	y_2
90	76·92	1	76·92	79·85	1	79·85	81·92	1	81·92	y_3
120	52·07	·722	37·59	53·33	·758	40·42	54·16	·780	42·24	y_4
150	41·66	·356	14·82	42·35	·392	16·60	42·51	·413	17·55	y_5
180	39·27	0	0	39·27	0	0	39·27	0	0	y_6
210	·75	− ·356	− ·27	·85	− ·392	− ·33	·90	− ·413	− ·37	y_7
240	3·54	− ·722	− 2·55	3·89	− ·758	− 2·95	4·13	− ·780	− 3·22	y_8
270	10·40	− 1	− 10·40	11·26	− 1	− 11·26	11·79	− 1	− 11·79	y_9
300	27·35	− 1·010	− 27·62	28·94	− ·974	− 28·18	29·93	− ·952	− 28·49	y_{10}
330	67·19	− ·644	− 43·27	68·85	− ·608	− 41·86	69·89	− ·586	− 40·95	y_{11}
360	106·5	0	0	106·5	0	0	106·5	0	0	—

TABLE IV.—CRANK EFFORT COEFFICIENTS.

Connecting Rod Length.	Cosine Terms.							Sine Terms.				
	a_0.	a_1.	a_2.	a_3.	a_4.	a_5.	a_6.	b_1.	b_2	b_3	b_4	b_5
3 cranks . .	30·7	24·3	− 10·6	− 12·7	− 13·9	− 11·6	− 6·1	64·6	48·6	25·5	11·9	4·4
4 cranks . .	30·7	23·1	− 11·4	− 12·4	− 13·5	− 10·8	− 5·7	65·4	47·0	23·8	10·9	4·2
5 cranks . .	30·7	22·4	− 12	− 12·2	− 13·2	− 10·2	− 4·7	65·6	45·9	22·7	10·2	3·8

For an engine having a rod four cranks long, we have $a_2 = -11{\cdot}4$, $b_2 = 47$, whence $A_2 = 48{\cdot}3$.

$$\therefore E_2 = 96{\cdot}6 \sin (2\theta + \epsilon).$$

The maximum value of E_2 is therefore 96·6 lbs. If the crank radius be 2 inches, this gives the maximum secondary explosion-torque as 16 lbs.-ft., the figures of course referring to an engine having each piston 1 square inch in area.

In the six-cylinder petrol engine we have $c = 3$, so that the third and sixth harmonics of E are unbalanced. The unbalanced third harmonic is

$$E_3 = 3(a_3 \cos 3\theta + b_3 \sin 3\theta) = 3A_3 \sin (3\theta + \epsilon),$$

where $A_3{}^2 = a_3{}^2 + b_3{}^2$. For a rod four cranks long we get

$$E_3 = 3 \times 27 \sin (3\theta + \epsilon).$$

The maximum value of E_3 is therefore 81 lbs. As the explosion-torque is r times the crank-effort, we have for a 2 inch crank a torque $13{\cdot}5 \sin (3\theta + \epsilon)$ occasioned by E_3. It must be remembered that these figures refer to an engine 1·128 in. bore. For a bore D inches the above values should be multiplied by $\cdot 7854D^2$.

The figures show that the four-cylinder engine does not compare unfavourably with the six-cylinder petrol motor as regards fluctuation of explosion torque.

We may note that the engine-frame torque due to explosion is equal and opposite to the crankshaft torque due to explosion.

§ 36. **Total Torque on Crankshaft and Engine-Frame.**—The total torque on the crankshaft is the sum of the explosion and inertia torques (due to both reciprocating mass and connecting rod). In the four-cylinder petrol engine, the largest harmonics of these torques are the secondaries, and it is of interest to note that that due to inertia in part counteracts that due to explosion. The secondary inertia-torque, due to a reciprocating mass (piston and equivalent mass at gudgeon pin due to rod) of ·4 lbs. per square inch of piston area πR^2, is (rod = 4 cranks), for a four-cylinder engine,

$$4\pi R^2(-\cdot 5001 \times \cdot 4r^2\omega^2 \sin 2\theta) \text{ poundals-ft.}$$

The secondary explosion-torque is

$$2\pi R^2 r(47 \sin 2\theta - 11{\cdot}4 \cos 2\theta) \text{ lbs.-ft.}$$

For $r = 2$ inches and $\omega \doteq 150$ radians per sec., we have the inertia-torque balancing the sine component of the explosion-torque.

In the six-cylinder petrol engine the largest fluctuation in torque is given by the third harmonic. With the above figures the tertiary inertia-torque is

$$6\pi R^2(-\cdot 1920 \times \cdot 4r^2\omega^2 \sin 3\theta) \text{ poundals-ft.}$$

The tertiary explosion-torque is

$$3\pi R^2 r(23\cdot 8 \sin 3\theta - 12\cdot 4 \cos 3\theta) \text{ lbs.-ft.}$$

For $r = 2$ inches and $\omega \doteq 172$ radians per sec., we get the sine component of the explosion-torque balancing the inertia-torque.

In both engines the torque on the engine frame is practically the same as that on the crankshaft, but opposite in direction. The slight difference between the total crankshaft and total engine-frame torque is due to the fact that the "equivalent rim" of rod gives different torques on the crankshaft and engine frame.

§ 37. **Note Regarding Couples on Engine-Frame.**—In the foregoing articles we have neglected consideration of the fact that the inertia-forces do not in general form coplanar systems. In other words, we have considered our engine to be condensed by end-wise contraction into a single plane, thus eliminating the effect of cylinder pitch. The effect of cylinder pitch in introducing pitching couples on the engine frame will be found discussed in Chapter XI. The rocking couples on the engine-frame due to inertia and explosion torques tend to vibrate the engine about an axis parallel to the crankshaft.

It is of importance to note that in the statement of the force-system acting on the engine-frames, §§ 20, 41, we have taken the force to act through the centre of the crankshaft bearing. Remembering that a force through the crankshaft bearing may (as far as the motion [vibration] of the engine is concerned), be replaced by an equal force acting through any other assigned point, and a couple, it will be apparent that the total "rocking couple" on the engine-frame will depend on the point assigned to the line of action of the force. By assigning the line of action of the force to pass through the centre of gravity of the complete engine, the disturbing effect of the force-system may be conveniently estimated.

CHAPTER IV.

DESAXÉ ENGINE.

A. *Inertia-Force.*

§ 38. **Piston Displacement, Velocity and Acceleration.**—In the desaxé or offset engine, the line of stroke of the wrist pin does not pass through the centre of the crankshaft. This may arise by

(*a*) Having the wrist pin displaced from the centre of the piston, the axis of cylinder passing through centre of crankshaft.
(*b*) The axis of the cylinder not passing through the centre of the crankshaft.
(*c*) Combination of (*a*) and (*b*).

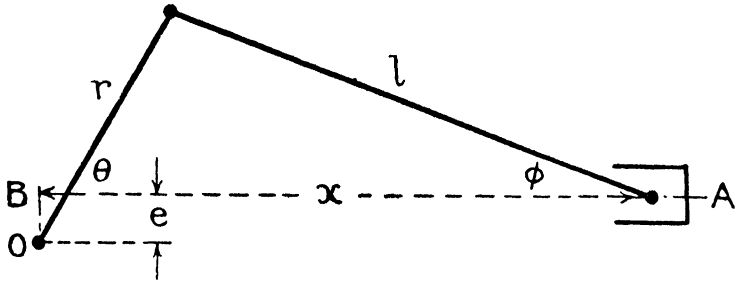

FIG. 13.—Desaxé engine.

The amount of offset met with, varies from $\frac{1}{5}$ to $\frac{1}{6}$th the crank radius. We shall throughout assume an offset equal to $\frac{1}{5}$th crank radius. In Fig. 13 the offset is denoted by *e*, AB is the line of stroke and O the centre of the crankshaft. By projection on the line of stroke we have

$$x = r \cos \theta + l \cos \phi = r(\cos \theta + \frac{1}{\rho} \cos \phi).$$

Now $\cos \phi = (1 - \sin^2 \phi)^{\frac{1}{2}}$, which, on expansion by the Binomial Theorem, becomes

$$1 - \tfrac{1}{2} \sin^2 \phi - \tfrac{1}{8} \sin^4 \phi - \tfrac{1}{16} \sin^6 \phi - \ldots \quad . \quad (1)$$

Referring again to the figure, we have

$$l \sin\phi = r \sin\theta - e.$$

$$\therefore \sin\phi = \rho \sin\theta - \gamma; \text{ where } \gamma = e/l.$$

On setting this value of $\sin\phi$ in (1), we get

$$\begin{aligned}
\cos\phi = 1 &- \tfrac{1}{2}(\rho^2 \sin^2\theta - 2\rho\gamma \sin\theta + \gamma^2) \\
&- \tfrac{1}{8}(\rho^4 \sin^4\theta - 4\rho^3\gamma \sin^3\theta + 6\rho^2\gamma^2 \sin^2\theta - 4\rho\gamma^3 \sin\theta + \gamma^4) \\
&- \tfrac{1}{16}(\rho^6 \sin^6\theta - 6\rho^5\gamma \sin^5\theta + 15\rho^4\gamma^2 \sin^4\theta - 20\rho^3\gamma^3 \sin^3\theta \\
&\qquad + 15\rho^2\gamma^4 \sin^2\theta - 6\rho\gamma^5 \sin\theta + \gamma^6) \\
&- \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&= 1 - \tfrac{1}{2}\gamma - \tfrac{1}{8}\gamma^4 - \tfrac{1}{16}\gamma^6 - \ . \ . \ . \\
&+ \rho(\gamma + \tfrac{1}{2}\gamma^3 + \tfrac{3}{8}\gamma^5 + \ . \ . \ .) \sin\theta \\
&+ \rho^2(- \tfrac{1}{2} - \tfrac{3}{4}\gamma^2 - \tfrac{15}{16}\gamma^4 - \ . \ . \ .) \sin^2\theta \\
&+ \rho^3(\tfrac{1}{2}\gamma + \tfrac{5}{4}\gamma^3 + \ . \ . \ .) \sin^3\theta \\
&+ \rho^4(- \tfrac{1}{8} - \tfrac{15}{16}\gamma^2 - \ . \ . \ .) \sin^4\theta \\
&+ \rho^5(\tfrac{3}{8}\gamma + \ . \ . \ .) \sin^5\theta \\
&+ \rho^6(- \tfrac{1}{16} - \ . \ . \ .) \sin^6\theta.
\end{aligned}$$

$$\begin{aligned}
\therefore \rho^{-1} \cos\phi = s_0 + s_1 \sin\theta + s_2 \sin^2\theta + s_3 \sin^3\theta + s_4 \sin^4\theta \\
+ s_5 \sin^5\theta + s_6 \sin^6\theta + \ . \ . \ . \quad . \quad . \quad . \qquad (2)
\end{aligned}$$

where

$$\begin{aligned}
s_0 &= \rho^{-1}(1 - \tfrac{1}{2}\gamma^2 - \tfrac{1}{8}\gamma^4 - \tfrac{1}{16}\gamma^6 - \ . \ . \ .) \\
s_1 &= \gamma + \tfrac{1}{2}\gamma^3 + \tfrac{3}{8}\gamma^5 + \ . \ . \ . \\
s_2 &= \rho(- \tfrac{1}{2} - \tfrac{3}{4}\gamma^2 - \tfrac{15}{16}\gamma^4 - \ . \ . \ .) \\
s_3 &= \rho^2(\tfrac{1}{2}\gamma + \tfrac{5}{4}\gamma^3 + \ . \ . \ .) \\
s_4 &= \rho^3(- \tfrac{1}{8} - \tfrac{15}{16}\gamma^2 - \ . \ . \ .) \\
s_5 &= \rho^4(\tfrac{3}{8}\gamma + \ . \ . \ .) \\
s_6 &= \rho^5(- \tfrac{1}{16} - \ . \ . \ .).
\end{aligned}$$

Writing (2) in exponential form, we get

$$\begin{aligned}
\frac{1}{\rho}\cos\phi = s_0 + \frac{s_1}{2i}(e^{i\theta} - e^{-i\theta}) + \frac{s_2}{-4}(e^{2i\theta} + {}^{-2i\theta} - 2) \\
+ \frac{s_3}{-8i}\left\{e^{3i\theta} - e^{3i\theta} - 3(e^{i\theta} - e^{-i\theta})\right\} \\
+ \frac{s_4}{16}\left\{e^{4i\theta} + e^{-4i\theta} - 4(e^{2i\theta} + e^{-2i\theta}) + 6\right\} \\
+ \frac{s_5}{32i}\left\{e^{5i\theta} - e^{-5i\theta} - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})\right\} \\
+ \frac{s_6}{-64}\left\{e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) \right. \\
\left. - 20\right\} + \ . \ . \ .
\end{aligned}$$

This on rearrangement gives

$$\begin{aligned}
\rho^{-1}\cos\phi = s_o &+ \tfrac{1}{2}s_2 + \tfrac{3}{8}s_4 + \tfrac{5}{16}s_6 + \ldots \\
&+ (s_1 + \tfrac{3}{4}s_3 + \tfrac{5}{8}s_5 + \ldots)\sin\theta \\
&+ (-\tfrac{1}{4}s_3 - \tfrac{5}{16}s_5 - \ldots)\sin 3\theta \\
&+ (\tfrac{1}{16}s_5 + \ldots)\sin 5\theta \\
&+ \ldots\ldots\ldots\ldots \\
&+ (-\tfrac{1}{2}s_2 - \tfrac{1}{2}s_4 - \tfrac{15}{32}s_6 - \ldots)\cos 2\theta \\
&+ (\tfrac{1}{8}s_4 + \tfrac{3}{16}s_6 + \ldots)\cos 4\theta \\
&+ (-\tfrac{1}{32}s_6 - \ldots)\cos 6\theta \\
= a_o &+ a_2\cos 2\theta + a_4\cos 4\theta + a_6\cos 6\theta + \ldots \\
&+ b_1\sin\theta + b_3\sin 3\theta + b_5\sin 5\theta + \ldots
\end{aligned}$$

the notation being obvious.

Hence the piston displacement x is

$$\begin{aligned}
x = r(\cos\theta &+ a_o + a_2\cos 2\theta + a_4\cos 4\theta + \ldots \\
&+ b_1\sin\theta + b_3\sin 3\theta + b_5\sin 5\theta + \ldots).
\end{aligned}$$

$$\begin{aligned}
\text{Piston velocity, } \frac{dx}{dt} &= -r\omega(\sin\theta + 2a_2\sin 2\theta + 4a_4\sin 4\theta + \ldots \\
&\quad - b_1\cos\theta - 3b_3\cos 3\theta - 5b_5\cos 5\theta - \ldots)
\end{aligned}$$

$$\begin{aligned}
\text{Piston acceleration, } \frac{d^2x}{dt^2} &= -r\omega^2(\cos\theta + 4a_2\cos 2\theta \\
&\quad + 16a_4\cos 4\theta + \ldots \\
&\quad + b_1\sin\theta + 9b_3\sin 3\theta + 25b_5\sin 5\theta + \ldots).
\end{aligned}$$

It will be observed that this expression for the acceleration, and consequently for the inertia-resistance of the piston, in the offset engine differs from that for the symmetrical engine, in that it contains odd harmonics as well as having the coefficients of the even harmonics slightly greater than the corresponding symmetrical-engine coefficients.

§ 39. **Balance of Inertia Forces: Multicylinder Offset Engine.**—The balance of the even harmonics is investigated in exactly the same manner as in Art. 18, and we arrive at the same general result, viz., those even harmonics which are integral multiples of the number of cranks are unbalanced. The investigation of the balance of the odd harmonics, or sine terms, is as given in Art. 23, and the result there found enables us to state that the harmonics, which are integer multiples of the number of cranks, are unbalanced. All other harmonics are in balance. It follows immediately from this fact that an engine having an even number of cranks appearing in end view has no unbalanced odd harmonic. Thus all odd harmonics of the four-cylinder petrol engine are in balance. The six-cylinder offset engine cranks at 120°, has the 3rd, 9th, 15th, . . . odd harmonics unbalanced. The unbalanced 3rd harmonic is $mr\omega^2 \, . \, 9b_3 \sin 3\theta$ per cylinder. The maximum value of this is (see Art. 40 for a rod four cranks long) $mr\omega^2 \times \cdot 0037$ per cylinder. The maximum unbalanced secondary inertia-force in the symmetrical four-cylinder petrol engine is $mr\omega^2 \times \cdot 2550$ per cylinder. Hence the offset six-cylinder has an unbalanced third harmonic approximately $\frac{1}{70}$ the amount of the unbalanced secondary in the four-cylinder. ($\cdot 2550 = p_2 = 4q_2$. See pp. 15, 143.)

§ 40. **Values of the Coefficients.**

Example 1.—Calculate s_0, s_1, s_2, s_3, s_4, s_5, s_6 for $e = \frac{1}{5}r$ and the following values of ρ :—

$\rho = \frac{1}{3}$; $\gamma = \frac{1}{15}$. Ans. : 2·9399 ; ·0668 ; — ·1678 ; ·0037 ;
— ·0048 ; ·0003 ; — ·00026.
$\rho = \frac{1}{4}$; $\gamma = \frac{1}{20}$. Ans. : 3·9199 ; ·0500 ; — ·1255 ; ·0016 ;
— ·0020 ; ·0001 ; — ·00006.
$\rho = \frac{1}{5}$; $\gamma = \frac{1}{25}$. Ans. : 4·8999 ; ·0400 ; — ·1002 ; ·0008 ;
— 0010 ; ·00003 ; — ·00002.

Example 2.—Calculate a_2, a_4, a_6, b_1, b_3, b_5 for the following values :

$\rho = \frac{1}{3}$; $\gamma = \frac{1}{15}$. Ans. : ·0864 ; — ·0006 ; ·000008 ; ·0698 ;
— ·0010 ; ·00002.
$\rho = \frac{1}{4}$; $\gamma = \frac{1}{20}$. Ans. : ·0638 ; — ·0003 ; —— ; ·0513 ;
— ·0004 ; ——.
$\rho = \frac{1}{5}$; $\gamma = \frac{1}{25}$. Ans. : ·0506 ; — ·0001 ; —— ; ·0406 ;
— ·0002 ; ——.

B. *Inertia-Torque.*

§ 41. **Effects of Inertia-Resistance of Piston.**—The inertia resistance, or "inertia pressure," developed by the acceleration of the reciprocating piston-mass, gives a torque on the crank-shaft and reactions on the engine-frame. If F be the inertia resistance of the piston, we have, from Fig. 14, the thrust S along the rod due to F given by

$$\mathrm{S} \cos \phi = \mathrm{F}.$$

The rod thrust S, when resolved into horizontal and vertical components at N, gives (1) a vertical component S sin ϕ, which appears as a vertical force on the crankshaft bearing; and (2) a horizontal force F, which gives a couple F . PN on the crankshaft, with a horizontal force F on the crankshaft bearing. The rod thrust S at B gives a downward load S $\sin\phi$ on the cylinder walls, so that, acting on the engine-frame, we have the vertical forces S $\sin\phi$ on the crankshaft bearing at P, and S $\sin\phi$ on the cylinder walls at B. These two forces give a couple S $\sin\phi$. BG on the engine-frame.

$$S\sin\phi \,.\, BG = F\sec.\phi\sin\phi \,.\, BG = F \,.\, BG\tan\phi = F \,.\, NG.$$

As NG = NP $- e$, the engine-frame torque is less than the crankshaft torque by the amount Fe. The horizontal force F at the crankshaft bearing has been investigated in Part A of this chapter. See note § 37 regarding position assigned to F.

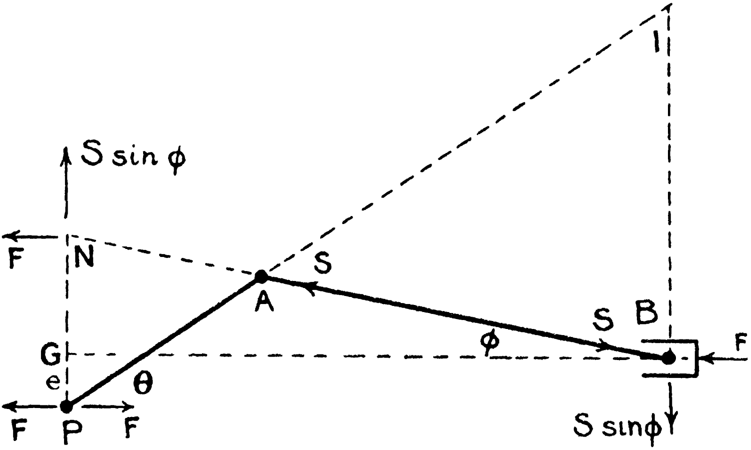

FIG. 14.—Desaxé engine: force analysis.

The instantaneous centre of rotation of the rod is I, so that we have

$$\text{Velocity of B/Velocity of A} = BI/AI = PN/PA.$$

$$\therefore\ v/\omega r = PN/r.$$

Hence, as in the symmetrical engine we get PN $= v/\omega$. The crankshaft torque T is therefore

$$T = F \,.\, v/\omega = mfv/\omega.$$

The engine-frame torque S is

$$S = mfe - mfv/\omega.$$

§ 42. **Series for Inertia-Torque.**—By taking the product of the series for f and v, given in Art. 38, we obtain the series

$$fv = r^2\omega^3(t_1 \sin \theta + t_2 \sin 2\theta + t_3 \sin 3\theta + t_4 \sin 4\theta + t_5 \sin 5\theta + t_6 \sin 6\theta + \ldots + u_1 \cos \theta + u_2 \cos 2\theta + u_3 \cos 3\theta + u_4 \cos 4\theta + u_5 \cos 5\theta + u_6 \cos 6\theta + \ldots).$$

From Art. 41, the inertia crankshaft torque is therefore

$$\mathrm{T} = mr^2\omega^2(t_1 \sin \theta + t_2 \sin 2\theta + t_3 \sin 3\theta + \ldots + u_1 \cos \theta + u_2 \cos 2\theta + \ldots),$$

and the inertia engine-frame torque is

$$\mathrm{S} = mer\omega^2(\cos \theta + 4a_2 \cos 2\theta + \ldots + b_1 \sin \theta + 9b_3 \sin 3\theta + \ldots) - mr^2\omega^2(t_1 \sin \theta + t_2 \sin 2\theta + \ldots + u_1 \cos \theta + u_2 \cos 2\theta + \ldots)$$

Example.—Calculate t_1, $-t_2$, $-t_3$, t_4, t_5, $-t_6$ for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: ·0864, ·4989, ·2631, ·0294, ·0065, ·0012
$\rho = \frac{1}{4}$. Ans.: ·0637, ·4989, ·1927, ·0160, ·0025, ·0003
$\rho = \frac{1}{5}$. Ans.: ·0506, ·4992, ·1525, ·0102, ·0012, ——.

Example.—Calculate u_1, u_2, u_3, u_4, u_5, u_6 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: ·0058, − ·0728, − ·0180, ·0062, ·0013, − ·0003
$\rho = \frac{1}{4}$. Ans.: ·0034, − ·0525, − ·0099, ·0025, ·0004, − ·0001
$\rho = \frac{1}{5}$. Ans.: ·0021, − ·0412, − ·0062, ·0012, ·0001, ——.

§ 43. **Balance of Inertia-Torque.**—As with the symmetrical engine we have balance of all harmonics other than those which are integral multiples of the number of cranks. For the crankshaft torque it will be observed that the coefficients of the important harmonics are much the same as for the symmetrical engine.

C. *Inertia-Torque due to Rod.*

§ 44. **Angular Velocity of Rod.**—From Fig. 13 we have

$$r \sin \theta = l \sin \phi + e.$$

$$\therefore r \cos \theta \frac{d\theta}{dt} = l \cos \phi \frac{d\phi}{dt}.$$

$$\therefore \frac{d\phi}{dt} = \rho\omega \cos \theta(1 - \sin^2 \phi)^{-\frac{1}{2}}.$$

This on expansion gives a series which may be written

$$\phi = \frac{d\phi}{dt} = \omega(e_1 \cos\theta + e_3 \cos 3\theta + e_5 \cos 5\theta + \ldots \\ + f_2 \sin 2\theta + f_4 \sin 4\theta + f_6 \sin 6\theta).$$

$$\therefore \ddot{\phi} = -\omega^2(e_1 \sin\theta + 3e_3 \sin 3\theta + 5e_5 \sin 5\theta + \ldots \\ - 2f_2 \cos 2\theta - 4f_4 \cos 4\theta - 6f_6 \cos 6\theta - \ldots).$$

It will be found that the sine coefficients are slightly larger than the corresponding coefficients for the symmetrical engine.

On squaring the series for $\dot{\phi}$ we get a series of the form

$$\dot{\phi}^2 = \omega^2(c_o + c_2 \cos 2\theta + c_4 \cos 4\theta + \ldots + d_1 \sin\theta + d_3 \sin 3\theta + \ldots)$$
$$\therefore 2\dot{\phi}\ddot{\phi} = -\omega^3(2c_2 \sin 2\theta + 4c_4 \sin 4\theta + \ldots - d_1 \cos\theta \\ - 3d_3 \cos 3\theta - \ldots).$$

§ 45. **Equivalent Dynamical System.**—As in Art. 25, the rod may be replaced by an equivalent dynamical system of two detached masses at the crank and gudgeon pins, together with a fictitious rim of moment of inertia I′. The detached masses are treated as in Art. 25. The " rim " occasions a crankshaft torque

$$\mathrm{C} = \mathrm{I}'\ddot{\phi}\dot{\phi}/\omega,$$

and an engine-frame torque

$$\mathrm{S} = -\mathrm{I}'\ddot{\phi} - \mathrm{I}'\ddot{\phi}\dot{\phi}/\omega,$$

as in the case of the symmetrical engine.

§ 46. **Balance of Inertia Torques due to Rod.**—The only portion of the inertia-torque to be considered here is that due to I′, since that due to the detached masses can be included in Art. 41. We have from Arts. 44, 45 :

$$\mathrm{C} = \mathrm{I}'\ddot{\phi}\dot{\phi}/\omega = -\mathrm{I}'\omega^2(c_2 \sin 2\theta + 2c_4 \sin 4\theta + \ldots \\ - \tfrac{1}{2}d_1 \cos\theta - \tfrac{3}{2}d_3 \cos 3\theta - \ldots)$$
$$\mathrm{S} = -\mathrm{I}'\ddot{\phi} - \mathrm{C} = \mathrm{I}'\omega^2(c_2 \sin 2\theta + 2c_4 \sin 4\theta + \ldots \\ - \tfrac{1}{2}d_1 \cos\theta - \tfrac{3}{2}d_3 \cos 3\theta - \ldots \\ + e_1 \sin\theta + 3e_3 \sin 3\theta + \ldots - 2f_2 \cos 2\theta - 4f_4 \cos 4\theta - \ldots).$$

In a multicylinder engine all harmonics of C and S are in balance, except such as are multiples of the number of cranks.

D. *Explosion-Torque.*

§ 47. **Indicator Diagram.**—As in Art. 29, Tables I., II., and III. were prepared. Table I. gives the percentage stroke completed for various crank angles, and also the piston velocity/crank-pin velocity corresponding to these angles. In Table III. we have given the crank efforts calculated for various crank angles. The formula

$$E = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_6 \cos 6\theta + b_1 \sin\theta + \ldots + b_5 \sin 5\theta$$

for crank effort was assumed, and the coefficients given in Table IV. were calculated from Table III., as in Art. 32.

TABLE I.

Crank Angle.	Rod 3 Cranks long.		Rod 4 Cranks long.		Rod 5 Cranks long.	
	Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\text{Crank Pin Velocity}}$.	Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\text{Crank Pin Velocity}}$.	Per Cent. Stroke.	$\frac{\text{Piston Velocity}}{\text{Crank Pin Velocity}}$.
0	·11	– ·067	·06	– ·05	·04	– ·04
30	7·12	·586	7·01	·565	6·95	·552
60	28·37	·977	27·52	·949	27·02	·932
90	55·00	1	53·75	1	53·00	1
120	78·37	·755	77·52	·783	77·02	·80
150	93·72	·414	93·61	·435	93·55	·447
180	100	·067	100	·05	100	·04
210	97·04	– ·299	96·11	– ·349	95·55	– ·378
240	84·13	– ·689	81·84	– ·733	80·48	– ·76
270	61·66	– 1	58·75	– 1	57·00	– 1
300	34·13	– 1·04	31·84	– ·999	30·48	– ·97
330	10·44	– ·701	9·51	– ·651	8·95	– ·620
360	·11	– ·067	·06	– ·05	·04	– ·04

TABLE II.

Crank Angle.	Rod 3 Cranks long.		Rod 4 Cranks long.		Rod 5 Cranks long.	
	Volume.	Pressure.	Volume.	Pressure.	Volume.	Pressure.
0	25·11	121	25·06	121	25·04	121
30	32·12	314	32·01	317	31·95	318
60	53·37	164	52·52	167	52·02	168
90	80·00	97	78·75	98	78·00	100
120	103·37	69	102·52	69·5	102·02	70
150	118·72	58	118·61	58	118·55	58
180	125	54	125	54	125	54
210	122·04	16	121·11	16	120·55	16
240	109·13	18	106·84	19	105·48	19
270	86·66	24	83·75	25	82	25
300	59·13	40	56·84	42	55·48	43
330	35·44	76	34·51	78	33·95	82
360	25·11	121	25·06	121	25·04	121

TABLE III.

Crank Angle.	Rod 3 Cranks long.			Rod 4 Cranks long.			Rod 5 Cranks long		
	Gauge Pressure.	$\frac{\text{Velocity}}{\omega r}$.	Crank Effort.	Gauge Pressure.	$\frac{\text{Velocity}}{\omega r}$.	Crank. Effort.	Gauge Pressure.	$\frac{\text{Velocity}}{\omega r}$.	Crank Effort.
0	106	- ·067	- 7	106	- ·05	- 5	106	- ·04	- 4
30	299	·586	174	302	·565	170	303	- ·552	167
60	149	·977	146	152	·949	144	153	·932	143
90	82	1	82	83	1	83	85	1	85
120	54	·755	41	54	·783	42	55	·80	44
150	43	·414	18	43	·435	19	43	·447	19
180	39	·067	3	39	·05	2	39	·04	1·5
210	1	- ·299	- ·3	1	- ·349	- .3	1	- ·378	- ·4
240	3	- ·689	- 2	4	- ·733	- 3	4	- ·76	- 3
270	9	- 1	- 9	10	- 1	- 10	10	- 1	- 10
300	25	- 1·04	- 26	27	- ·999	- 27	28	- ·97	- 27
330	61	- ·707	- 43	63	- ·651	- 41	67	- ·620	- 41

TABLE IV.—CRANK EFFORT COEFFICIENTS.

Connecting Rod Length.	a_0	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5
3 cranks .	31·4	21·4	- 13·7	- 15·2	- 14·1	- 11·3	- 5·5	65·8	47·3	24	10	3·7
4 cranks .	31·1	21·3	- 13·3	- 14·1	- 13·6	- 10·6	- 5·6	65·9	45·9	22·9	9·5	3·5
5 cranks .	31·1	20·8	- 13·9	- 13·7	- 13	- 10·1	- 5·5	66·1	45	22·1	9·5	3·4

§ 48. **Crank-Effort : Multicylinder Engine.**—It will be noted that the coefficients in Table IV. are not very different from the coefficients of crank-effort for the symmetrical engine. The Arts. 35, 36, 37, relating to the symmetrical engine, apply also to the offset engine. The engine-frame torque due to explosion is equal to the crank-torque, but opposite in direction. For let F (Fig. 14) be the fluid pressure on the piston. Then as in Art. 41 we get due to F, a force F on the crankshaft-bearing, a couple F . PN on the crankshaft, and a couple F . NG on the engine-frame. But we have also a fluid pressure F on the cylinder-head opposite to the force F on the piston. This pressure on the cylinder-head with the pressure F on the crankshaft-bearing gives a couple F . GP on the engine-frame, so that the ultimate effect of the fluid pressure in the cylinder is to give a couple F(NG + GP) on the engine-frame, and an equal and opposite couple F . NP on the crankshaft.

CHAPTER V.

RADIAL ENGINES.

Inertia Forces.

§ 49. **Examples of Radial Engines.**—In this type of engine the cylinders do not all lie in the same longitudinal plane. The opposed twin is the simplest example of a radial engine. The Brotherhood steam engine, with cylinders at 120°, is an equally simple illustration. Finally, we have many examples from aero engine designs, such as the Salmson, Anzani, Bristol, etc. We shall also discuss here engines of the **V**-type, such as the eight-cylinder **V** consisting of two sets of four cylinders set at an angle of 90°.

§ 50. **Inertia Forces.**—Considering a radial engine of c cylinders, the cylinders will have their centre lines make an angle λ with each other, as in Fig. 15, and we shall have $c\lambda = 2\pi$. The first four pistons are shown by the heavy dots, with their rods working on the crank, which makes an angle θ with the vertical or centre line of cylinder No. 0. The other rods, shown dotted in the figure, also work on this crank. Examining the third cylinder, we find the crank making an angle $\theta - 3\lambda$ with the line of stroke of this cylinder. The magnitude of the nth harmonic inertia-force developed in Cylinder No. 3 at the instant shown is therefore, from Art. 17.

$$mr\omega^2 p_n \cos n(\theta - 3\lambda) = \tfrac{1}{2}mr\omega^2 p_n \{e^{ni(\theta - 3\lambda)} + e^{-ni(\theta - 3\lambda)}\}.$$

This force, being along the centre line of Cylinder No. 3, makes an angle 3λ with the vertical; and taking the vertical to be our standard direction, the above force is completely represented by the vector

$$\tfrac{1}{2}mr\omega^2 p_n \{e^{ni(\theta - 3\lambda)} + e^{-ni(\theta - 3\lambda)}\}e^{3i\lambda}.$$

The inertia forces of the nth order due to the several cylinders forming the engine, will therefore appear as follows, omitting for convenience the constant factor $\frac{1}{2}mr\omega^2 p_n$:

Cylinder 0,	$\{e^{ni\theta} + e^{-ni\theta}\}.$
Cylinder 1,	$\{e^{ni(\theta-\lambda)} + e^{-ni(\theta-\lambda)}\}e^{i\lambda}.$
Cylinder 2,	$\{e^{ni(\theta-2\lambda)} + e^{-ni(\theta-2\lambda)}\}e^{2i\lambda}.$
. . .	. . . .
Cylinder $c-1$,	$\{e^{ni(\theta-\overline{c-1}\lambda)} + e^{-ni(\theta-\overline{c-1}\lambda)}\}e^{\overline{c-1}i\lambda}.$

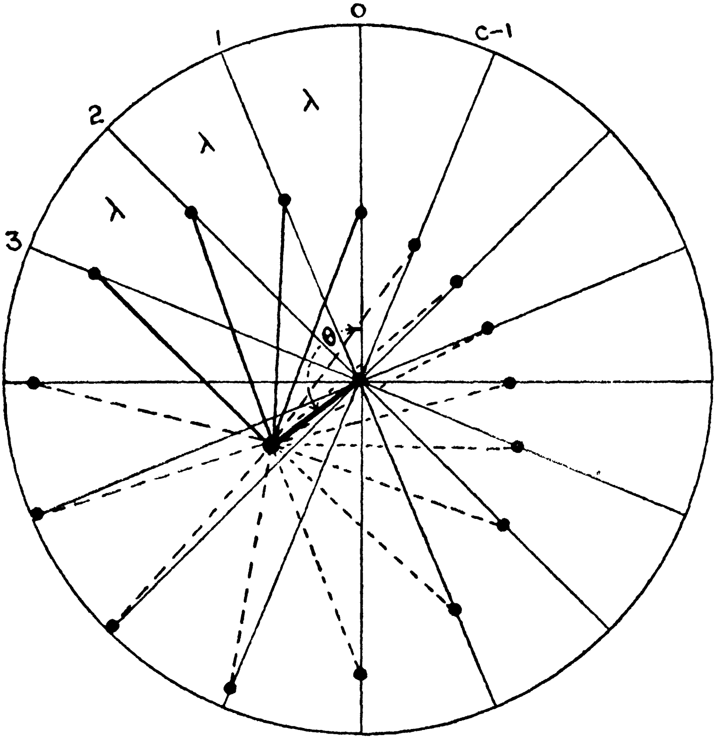

FIG. 15.—Radial engine. Cylindres fixes en étoile.

Summing, we get the total *n*th harmonic to be

$$
\begin{aligned}
F_n = {} & e^{ni\theta} + e^{-ni\theta} \\
& + e^{ni\theta-(n-1)i\lambda} + e^{-ni\theta+(n+1)i\lambda} \\
& + e^{ni\theta-(n-1)2i\lambda} + e^{-ni\theta+(n+1)2i\lambda} \\
& + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
& + e^{ni\theta-(n-1)(c-1)i\lambda} + e^{-ni\theta+(n+1)(c-1)i\lambda} \\
= {} & e^{-ni\theta}\{1 + e^{(n+1)i\lambda} + e^{(n+1)2i\lambda} + \ldots + e^{(n+1)(c-1)i\lambda}\} \\
& + e^{ni\theta} \,.\, e^{-(n-1)(c-1)i\lambda}\{1 + e^{(n-1)i\lambda} + e^{(n-1)2i\lambda} \\
& \qquad + \ldots + e^{(n-1)(c-1)i\lambda}\}.
\end{aligned}
$$

To have F_n zero for all values of θ, both bracketed series in the expression on the right must be separately zero. Now the series with index $n - 1$ is zero unless $n = 1$, or $n - 1$ be a multiple of the number of cylinders. The series with index $n + 1$ is zero unless $n + 1$ be an integral multiple of the number of cylinders. This may be proved as in Art. 18.

Therefore if $n - 1$ or $n + 1$ be an integral multiple of the number of cylinders, or if $n = 1$, the inertia forces of the nth order are not in balance. All other harmonics are in balance.

The relation $n \pm 1 = kc$ gives the unbalanced harmonics. This is equivalent to $n = kc \pm 1$. From this, if c be even, n is odd, hence only odd harmonics are unbalanced, and since no odd harmonics exist in the engine with line of stroke passing through centre of crankshaft, such radial engines with an even number of cylinders have all inertia force harmonics higher than the first in balance.

§ 51. **Nature of Unbalanced Harmonics.**—Referring to the expression for F_n, given in Art. 50, when $n = 1$ and $c > 2$, the series with index $n + 1$ is zero, while each term in the series having $n - 1$ as index is unity, so that its sum is c. We therefore have

$$F_1 = ce^{i\theta}.$$

F_1 is therefore a force represented by the vector $ce^{i\theta}$, that is, a force along the crank. The actual value of F_1, of course, is $\frac{1}{2}mr\omega^2 \cdot c$. Obviously, this force can be balanced by a suitable mass on the crankshaft. (In the Jupiter engine balance masses are arranged to throw their inertia (centrifugal) load on to the outer surface of the big end of the master rod, thus relieving the crank-pin bearing of inertia loading. This is effected by suspending the bob weights on links, which extend from the big end along the crank cheeks to the opposite side of the crankshaft. The bob weights fit in slots cut in the main crankshaft balance weights. The other ends of the links are furnished with white metalled bearings which strap over the outer surface of the big end of the master rod.)

In the other cases in which F_n does not vanish, it is readily seen that

$$F_n = ce^{\pm ni\theta},$$

which is a force represented by a vector rotating with ± n times the speed of the crank. It is only when $n = 1$ that it is possible to balance this by a mass simply rotating with the crankshaft.

An exceptional case is the primary inertia force of the opposed twin cylinder engine, that is, for $c = 2$. Here both the $n - 1$ and $n + 1$ series have each term unity, so that we get

$$F_1 = 2(e^{i\theta} + e^{-i\theta}) = 4 \cos \theta.$$

The actual value of the primary force here is therefore $\frac{1}{2}mr\omega^2 \, . \, 4 \cos \theta$, clearly twice the primary force of one cylinder.

§ 52. **Offset in Radial Engine.**—The Anzani engines are offset to the extent of approximately $\frac{1}{2}$ inch, that is, the cylinder centre lines, instead of radiating from a point, are tangential to a circle of $\frac{1}{2}$-inch radius (Fig. 24). In the Salmson engine the setting of the crank-pin cage is in effect equivalent to offsetting the cylinders. Fig. 16 shows the crank-pin cage for a seven-cylinder Salmson. With No. 0 cylinder vertical, the crank will have advanced through an angle θ_1 before the maximum load comes on the rod. The cage C carrying the crank-pins is constrained by epicyclic gearing to remain parallel to itself while its centre is carried round on the crank-pin. The arrow shows No. 0 rod when the thrust along it is a maximum, in which position it makes an angle ϕ_1 with the vertical. The crank-pin cage is set with radius to pin No. 0, making an angle $- \phi_1$ (about 6° for the seven-cylinder engine) with the vertical, so that the maximum rod-thrust will give no turning moment on the cage. The epicyclic train is thus not called upon to withstand undue strain. Draw OQ equal and parallel to PG. It will be evident that Q is a fixed point for all crank positions, and that G describes a circle about Q as centre. Thus the line of stroke of cylinder No. 0 does not pass through the centre of the circle described by the auxiliary crank-pin, but passes to the left of that point as in the ordinary offset engine.

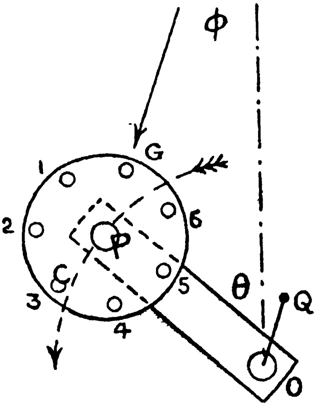

FIG. 16.—Crank-pin cage of Salmson seven-cylinder radial engine.

§ 53. **Odd Harmonics in Offset Engine.**—As the even harmonics but differ from the even harmonics of the symmetrical engine in that they are slightly larger, Art. 50 will apply to the balance of these harmonics in the offset radial. We have, however, to discuss the balance of the odd harmonics or sine terms which occur in the expression for the inertia force of the offset engine given in Art. 38. As in Art. 50, we have the following table for the *n*th odd harmonic :

Cylinder 0, $\frac{1}{i}\{e^{ni\theta} - e^{-ni\theta}\}$
Cylinder 1, $\frac{1}{i}\{e^{ni(\theta-\lambda)} - e^{-ni(\theta-\lambda)}\}e^{i\lambda}$
Cylinder 2, $\frac{1}{i}\{e^{ni(\theta-2\lambda)} - e^{-ni(\theta-2\lambda)}\}e^{2i\lambda}$
. . . .
Cylinder $c - 1$, $\frac{1}{i}\{e^{ni(\theta-\overline{c-1}\lambda)} - e^{-ni(\theta-\overline{c-1}\lambda)}\}e^{\overline{c-1}i\lambda}$.

Summing, we get

$$\begin{aligned} iF_n' = &- e^{-ni\theta}\{1 + e^{(n+1)i\lambda} + e^{(n+1)2i\lambda} + \ldots\} \\ &+ e^{ni\theta} \quad e^{-(n-1)(c-1)i\lambda}\{1 + e^{(n-1)i\lambda} + e^{(n-1)2i\lambda} + \ldots\}. \end{aligned}$$

Hence, when F_n' is not zero, that is, when $n \pm 1$ is a multiple of c, the value of F_n' is

$$F_n' = \frac{ce^{ni\theta}}{i} = - ice^{ni\theta} \text{ (for } n = 1 \text{ or } n - 1 = kc),$$

which is a force lagging 90° behind the direction of a crank rotating at n times the speed of the engine.

When $n + 1 = kc$ the value of F_n' is

$$F_n' = \frac{- ce^{-ni\theta}}{i} = ice^{-ni\theta}$$

which is a force whose direction is 90° behind the direction of a crank rotating at n times the engine speed, but in the reverse sense.

§ 54. **Primary Inertia-Force : Offset Engine.**—When $n = 1$ we have both a sine and a cosine term of the series for the offset engine unbalanced. The former we have just seen, gives a force whose direction rotates with the engine speed, and lags 90° behind the crank. The latter gives a force whose direction also rotates with engine speed, but has its line of action along the crank. The balance mass on the crankshaft of such an engine is placed opposite the resultant of these two forces, and is therefore less than 180° ahead of the crank. Numerically, we have $F_1 = \frac{1}{2}mr\omega^2c$ and $F_1' = \frac{1}{2}mr\omega^2c \cdot {\cdot}0513$ for rod four cranks long, and offset one-fifth crank radius. The resultant of F_1 and F_1', therefore, makes an angle $\tan^{-1} {\cdot}0513$ with the direction of F_1. This is almost 3°, so that the balance masses would be 177° ahead of the crank in such an engine.

V-*Engines.*

§ 55. **Inertia-Force: Twin-Cylinder V at Λ.**—Taking our standard direction along the axis of one of the cylinders, the *n*th harmonic of that cylinder is, from Art. 17

$$mr\omega^2 p_n \cos n\theta$$

where θ is the angle made by the crank with the line of stroke (Fig. 17). The *n*th harmonic due to the second cylinder is

$$mr\omega^2 p_n \cos n(\theta - \lambda)$$

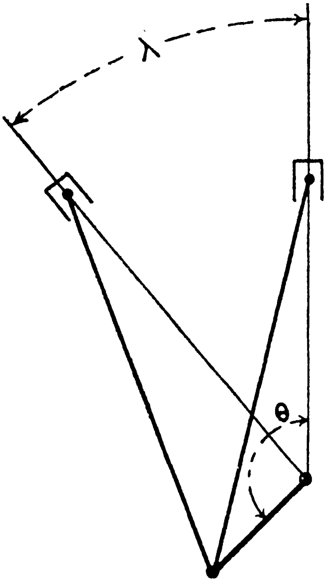

FIG. 17.—Cylinders V at angle λ.

along the axis of that cylinder. This force is represented, therefore, by the vector

$$mr\omega^2 p_n \cos n(\theta - \lambda) e^{i\lambda}.$$

The total *n*th harmonic is, therefore (omitting the constant factor $\frac{1}{2}mr\omega^2 p_n$),

$$\begin{aligned}
F_n &= e^{ni\theta} + e^{-ni\theta} + (e^{ni\overline{\theta - \lambda}} + e^{-ni\overline{\theta - \lambda}})e^{i\lambda} \\
&= e^{ni\theta}\{1 + e^{(1-n)i\lambda}\} + e^{-ni\theta}\{1 + e^{(1+n)i\lambda}\} \\
&= e^{\frac{1}{2}(1-n)i\lambda} \,.\, e^{ni\theta}\{e^{\frac{1}{2}(1-n)i\lambda} + e^{-\frac{1}{2}(1-n)i\lambda}\} \\
&\qquad + e^{\frac{1}{2}(1+n)i\lambda} \,.\, e^{-ni\theta}\{e^{\frac{1}{2}(1+n)i\lambda} + e^{-\frac{1}{2}(1+n)i\lambda}\} \\
&= e^{\frac{1}{2}(1-n)i\lambda} \,.\, e^{ni\theta} \,.\, 2\cos\tfrac{1}{2}(n - 1)\lambda \\
&\qquad + e^{\frac{1}{2}(1+n)i\lambda} \,.\, e^{-ni\theta} \,.\, 2\cos\tfrac{1}{2}(n + 1)\lambda.
\end{aligned}$$

The first term represents a constant force $2 \cos \frac{1}{2}(n - 1)\lambda$, rotating at n times the crank speed, the factor $e^{\frac{1}{2}(1-n)i\lambda}$ giving the angle by which the direction is in advance of the angle $n\theta$. The second term represents a constant force, $2 \cos \frac{1}{2}(n + 1)\lambda$, rotating at $- n$ times the crank speed, the factor $e^{\frac{1}{2}(1+n)i\lambda}$ giving the angle of lag of its direction behind the angle $- n\theta$.

When $n = 1$ we have

$$F_1 = 2e^{i\theta} + 2e^{i\lambda} \, . \, e^{-i\theta} \cos \lambda.$$

The first term here represents a force rotating with the crank. The second term represents a constant force rotating in the opposite direction with same speed as crank, the direction of the force lagging by the angle λ behind the angle $- \theta$. Putting $\lambda = 180°$, we get the primary inertia force of the opposed twin :

$$F_1 = 2e^{i\theta} + 2e^{-i\theta} = 4 \cos \theta.$$

With the constant factor which we have omitted to write, this becomes $\frac{1}{2}mr\omega^2 \, . \, 4 \cos \theta$, a result which can be readily verified from elementary analysis.

§ 56. **Twin-cylinder V at 90°.**—Assuming one of the cylinders vertical, its nth harmonic inertia-force is $mr\omega^2 p_n \cos n\theta$. The nth harmonic of the second cylinder is $mr\omega^2 p_n \cos n(\theta - 90°) \times i$, for the direction of this force makes 90° with our standard direction. We therefore have (omitting the constant factor $mr\omega^2 p_n$),

$$\begin{aligned} F_n &= \cos n\theta + i \cos n(\theta - 90°) \\ &= \cos n\theta + i \cos n\theta \text{ when } n \text{ is a multiple of } 4 \, ; \\ \text{or} \quad &= \cos n\theta - i \cos n\theta \text{ when } n \text{ is a multiple of } 2. \end{aligned}$$

In both cases F_n is the resultant of two equal forces at right angles, the magnitude of which is $\sqrt{2} \cos n\theta$. When n is a multiple of 4, F_n is represented by two equal forces along OA and OB (Fig. 18), so that F_n is a force $\sqrt{2} \cos n\theta$ along OC. When n is but a multiple of 2, F_n is represented by two equal forces along OA and BO, so that F_n is a force $\sqrt{2} \cos n\theta$ along OD. F_n is therefore identical with the inertia resistance of a mass having a simple harmonic motion along OD or OC. When $n = 1$,

$$\begin{aligned} F_1 &= \cos \theta + i \cos (\theta - 90) \\ &= \cos \theta + i \sin \theta \\ &= e^{i\theta}, \end{aligned}$$

which is a force along the crank, and, obviously, may be balanced by a suitable mass on the crankshaft opposite the crank-pin.

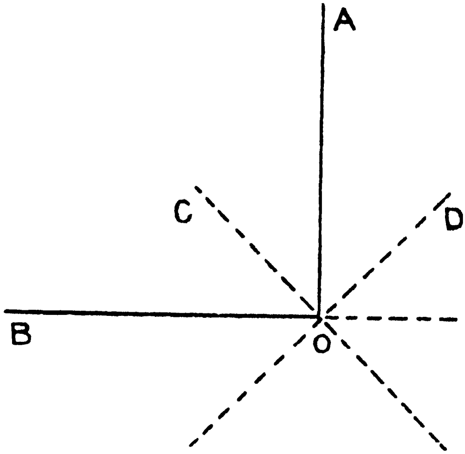

FIG. 18.—Cylinders V at 90°.

§ 57. **Eight-cylinder V at 90°.**—The engine, consisting of two sets of four cylinders placed at right angles, and working on a single four-throw crankshaft (Fig. 41), is shown in Fig. 19. Since each set of four cylinders has its primary forces balanced,

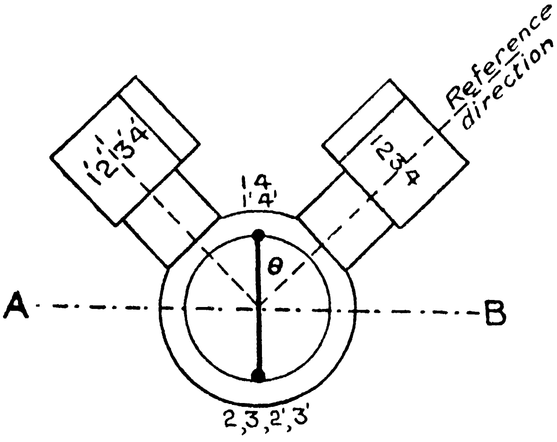

FIG. 19.—Eight-cylinder engine V at 90°.

the complete engine is in primary balance. The secondary force due to cylinders 1, 1′; 4, 4′, is (being two pairs of twins at 90°):

$$f_2 = 2(\cos 2\theta - i \cos 2\theta).$$

The secondary force due to the two twin pairs 2, 2′; 3, 3′, whose crank makes $180 + \theta$ with the standard direction, is

$$\begin{aligned} f_2' &= 2\{\cos 2(180 + \theta) - i \cos 2(180 + \theta)\} \\ &= 2(\cos 2\theta - i \cos 2\theta). \end{aligned}$$

The total secondary force is therefore

$$F_2 = f_2 + f_2' = 4(\cos 2\theta - i \cos 2\theta),$$

that is, a force $4\sqrt{2} \cos 2\theta$ along AB.

It will be observed that if the cranks 2, 3, 2′, 3′ made 90° with the cranks 1, 4, 1′, 4′, we would have

$$\begin{aligned} f_2' &= 2\{\cos 2(90 + \theta) - i \cos 2(90 + \theta)\} \\ &= -2\{\cos 2\theta - i \cos 2\theta\}, \end{aligned}$$

so that $F_2 = f_2 + f_2' = 0$. The secondary forces would, therefore, be in balance. As the primary force of each pair of **V**-cylinders could be balanced by a rotating mass on the crankshaft, the complete engine could be put into primary balance by suitable masses placed on the crankshaft. Another arrangement whereby primary and secondary balance may be secured is to place cranks 1 and 4 at 180° with each other; cranks 2 and 3 also at 180° with each other, and in a plane making 90° with the plane of 1 and 4.

The Napier "Cub" Aero engine consists of two sets of eight cylinders **V** at 90°, the sets making $52\frac{1}{2}°$ with each other as in Fig. 20. A single crankshaft of the ordinary four-cylinder petrol engine type (Fig. 41) is employed. The secondary inertia force due to the left set is $4\sqrt{2} \cos 2\theta$ along AB. At the instant the crank makes θ with cylinder-set 1, it makes $\theta - 217\frac{1}{2}°$ with cylinder-set 3, so that the secondary due to the sets 3 and 4 is a force $4\sqrt{2} \cos 2(\theta - 217\frac{1}{2}°)$ along A′B′. The secondary for the complete engine is the resultant of these two forces, and may be written as a vertical force,

$$e^{-5i\pi/48} \,.\, 4\sqrt{2} \cos 2\theta - e^{5i\pi/48} \,.\, 4\sqrt{2} \cos(2\theta - 75°),$$

and evaluated graphically.

§ 58. **Fan Type Engines.**—The Napier "Lion" Aero engine has twelve cylinders arranged in three blocks of four each. The central set (Fig. 21) is vertical, and works on the usual four-cylinder crankshaft. The other two sets of cylinders make 60° and − 60° with the vertical, their rods sharing the same crank-pin as the corresponding cylinder of the vertical set.

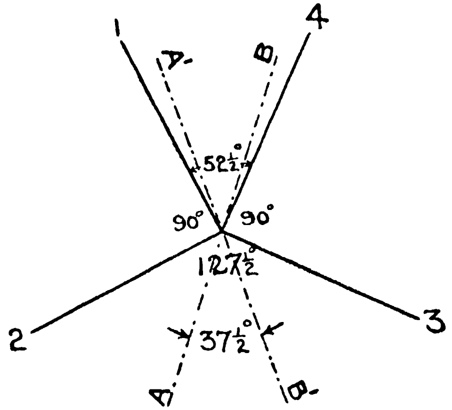

FIG. 20.—Napier "Cub" Areo engine.

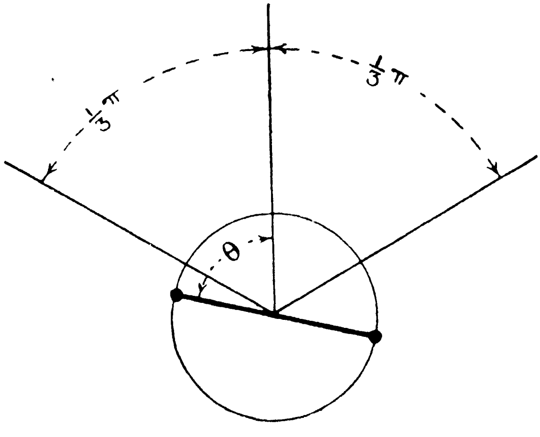

FIG. 21.—Fan arrangement of cylinders.

From Art. 19 we know that each row of four cylinders has an unbalanced nth-harmonic inertia-force, given by

$$4mr\omega^2 p_n \cos n\theta,$$

where θ is the angle made by the crank with the centre line of cylinders. Measuring θ from the vertical, we find the crank making $\theta - \frac{1}{3}\pi$ and $\theta + \frac{1}{3}\pi$ with the left and right row of cylinders respectively. The total nth harmonic is therefore :

Vertical set, $4mr\omega^2 p_n \cos n\theta$.
Left set, $e^{\frac{i\pi}{3}}\{4mr\omega^2 p_n \cos n(\theta - \frac{1}{3}\pi)\}$.
Right set, $e^{-\frac{i\pi}{3}}\{4mr\omega^2 p_n \cos n(\theta + \frac{1}{3}\pi)\}$.

We multiply the force due to the left set by the cyclic factor $e^{\frac{i\pi}{3}}$, since the direction of this force makes 60° with the vertical. Similarly we multiply the force due to the right set by the cyclic factor $e^{-\frac{i\pi}{3}}$, since its direction makes − 60° with the vertical. Adding now, we get the nth harmonic of the complete engine given by

$$\begin{aligned}
F_n &= 2mr\omega^2 p_n\{e^{ni\theta} + e^{-ni\theta} + e^{\frac{i\pi}{3}}(e^{ni(\theta - \frac{1}{3}\pi)} + e^{-ni(\theta - \frac{1}{3}\pi)}) \\
&\qquad + e^{-\frac{i\pi}{3}}(e^{ni(\theta + \frac{1}{3}\pi)} + e^{-ni(\theta + \frac{1}{3}\pi)})\} \\
&= 2mr\omega^2 p_n\Big[e^{ni\theta}\Big\{1 + e^{(n-1)\frac{i\pi}{3}} + e^{-(n-1)\frac{i\pi}{3}}\Big\} \\
&\qquad + e^{-ni\theta}\Big\{1 + e^{(n+1)\frac{i\pi}{3}} + e^{-(n+1)\frac{i\pi}{3}}\Big\}\Big] \\
&= 2mr\omega^2 p_n[e^{ni\theta}\{1 + 2\cos(n-1)\tfrac{\pi}{3}\} + e^{-ni\theta}\{1 + 2\cos(n+1)\tfrac{\pi}{3}\}].
\end{aligned}$$

Taking $n = 2$, this becomes

$$F_2 = 4mr\omega^2 p_2\, e^{2i\theta} - 2mr\omega^2 p_2\, e^{-2i\theta}.$$

The first term is a force $4mr\omega^2 p_2$ whose direction rotates at twice the crank speed, while the second term represents a force $2mr\omega^2 p_2$, whose direction rotates at twice the crank speed, but in the opposite sense. Plainly, the secondary is a maximum when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. F_2 then becomes

$$F_2 = 4mr\omega^2 p_2 i + 2mr\omega^2 p_2 i = i(6mr\omega^2 p_2),$$

that is, a horizontal force $6mr\omega^2 p_2$.

Otherwise, F_2 may be written

$$\begin{aligned}
F_2 &= 2mr\omega^2 p_2\{e^{2i\theta} + e^{2i\theta} - e^{-2i\theta}\} \\
&= 2mr\omega^2 p_2 e^{2i\theta} + i4mr\omega^2 p_2 \sin 2\theta,
\end{aligned}$$

that is, the secondary inertia force of this engine is equivalent to that due to a mass rotating at twice the crank speed, and a mass having a simple harmonic oscillation along a horizontal line. Of course, the primary inertia force of the engine is in balance, since each set separately has its primary in balance. The nature of the harmonics higher than the second may be examined as above. For example, putting $n = 4$ in the expression for F_n, we get

$$F_4 = 4mr\omega^2 p_4 e^{-4i\theta} - 2mr\omega^2 p_4 e^{4i\theta}.$$

We may note here that the inertia and explosion torques of this engine are the same as those of the twelve-cylinder **V** at 60°, given in Art. 61.

B. *Inertia-Torque due to Piston and to Rod.*

§ 59. **Inertia-Torque Same as for Cylinder-in-Line Type.**—The inertia-torque of a c-cylinder radial engine is identical with the inertia-torque of a c-cylinder in line type. That this is so, will be evident when we reflect that the inertia-torque for any cylinder depends only on the angle between the crank and the line of stroke of that cylinder, and not at all on the direction of the line of stroke. Imagine the crank of the radial engine to be composed of c superimposed cranks, each crank being identified with one of the c cylinders. The inertia-torque of any cylinder will not be altered if we rotate the whole mechanism of that cylinder until its line of stroke becomes vertical. If we do this for each cylinder of the radial engine, we obtain a c-crank, cylinder-in-line engine. Sections B and C of Chapters III. and IV., therefore, also apply to the radial engine.

§ 60. **Inertia-Torque : Eight-cylinder V at 90°.**—Taking the engine as consisting of two sets of four-cylinder engines working on a single four-throw crank of the usual four-cylinder petrol engine type (Fig. 41), let the set 1′, 2′, 3′, 4′ be rotated through 90° until these cylinders coincide with the cylinders of the other set, 1, 2, 3, 4 (Fig. 19). The cranks associated with the cylinders 1′, 2′, 3′, 4′, which originally coincided with the cranks 1, 2, 3, 4, will now make 90° with them, so that there results a four-crank-at-90° cylinder-in-line engine. Referring to Art. 23, the unbalanced inertia-torque of this engine consists of the 4th, 8th, 12th, etc., harmonics. For an engine consisting of two pairs of the four-cylinder of Art. 24, the total fourth harmonic has a maximum value

$$\frac{8 \times 3{\cdot}2 \times {\cdot}25^2 \times 200^2 \times {\cdot}0162}{32} = 32 \text{ lbs.-ft.}$$

It follows also from the above that the explosion-torque of this engine is the same as that of a four-crank-at-90° ; the largest unbalanced harmonic is the 4th. For a rod four cranks long, this torque is (Table IV., p. 37),

$$4\pi R^2 r(-13{\cdot}5\cos 4\theta + 10{\cdot}9\sin 4\theta)\ \text{lbs.-ft.}$$
$$= 4\pi R^2 r\ .\ 17{\cdot}3\sin(4\theta + \epsilon)\ \text{lbs.-ft.}$$

The explosion-torque of the Napier "Cub" is that of a sixteen-cylinder-in-line type working on two sets of four cranks at 90°, one set of cranks being $52\frac{1}{2}°$ in advance of the other set. The unbalanced fourth harmonic is

$$4\pi R^2 r\{a_4\cos 4\theta + b_4\sin 4\theta + a_4\cos 4(\theta + 52\tfrac{1}{2}°) + b_4\sin 4(\theta + 52\tfrac{1}{2}°)\}.$$

Note that if the angle $52\frac{1}{2}°$ were replaced by 45° this harmonic would be in balance.

§ 61. **Inertia-Torque: Twelve-cylinder V at 60°.**—An end view of this engine shows cranks at 120°. Rotating one set of six cylinders into coincidence with the other set will rotate their associated cranks through 60°, so that we get a cylinder-in-line engine with cranks at 60°. Reference to Art. 23 shows that the unbalanced inertia-torque harmonics are the 6th, 12th, 18th, etc. This engine is exceptionally good as regards inertia-torque, the coefficients of the unbalanced harmonics being very small. The explosion-torque is the same as that of a twelve-cylinder-in-line engine with cranks at 60°; the largest unbalanced harmonic is the 6th.

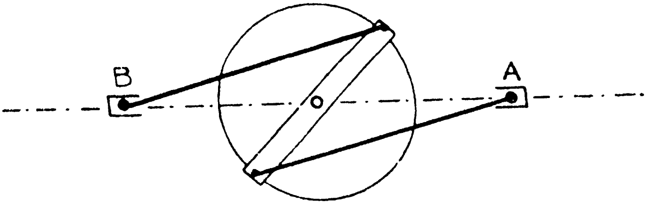

Fig. 22.—Opposed twin; cranks at 180°.

§ 62. **Radial Engines with Two Cranks at 180°.**—In the engine (Fig. 22) it will be evident that at any instant the acceleration of the piston A is equal and opposite to that of B, so that the inertia forces are completely balanced. The inertia torque due both to rods and reciprocating masses act in the same direction, and are thus equal to twice the inertia-torque due to one cylinder. If the cylinders be offset, as in Fig. 23, the inertia forces (which occasion horizontal loads on the crankshaft bearing as set out in Fig. 14) are at any instant equal and opposite, and therefore in complete balance. The inertia-torque due both to rods and reciprocating masses act in the same direction, and are thus equal to twice the inertia-torque due to one cylinder.

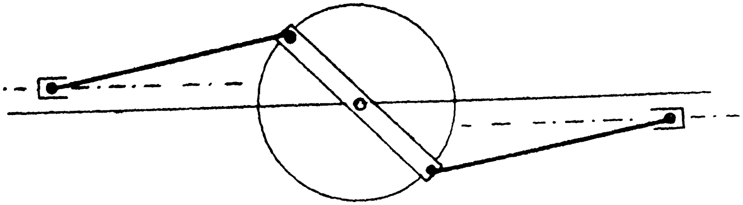

FIG. 23.—Offset opposed twin.

The six-cylinder Anzani radial engine (Fig. 24) consists of three sets of opposed cylinders, the axes of the cylinders making 60° with each other. The ten-cylinder Anzani consists of five

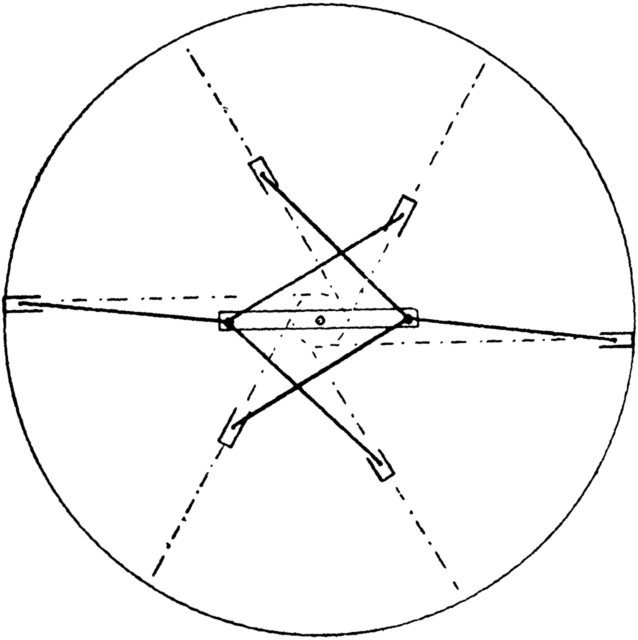

FIG. 24.—Six-cylinder Anzani radial engine.

such sets with their axes making 36° with each other. Plainly, the inertia forces are completely balanced while the inertia torque is the same as that due to the corresponding offset six (or ten) cylinder-in-line engine. We may note, however, that as the cranks are not coplanar, a pitching couple is introduced, the moment of which is the distance between planes of cranks multiplied by the inertia-force due to pistons working on either crank-pin. In the twenty-cylinder Anzani, the cranks are at 162°, while the cylinders are spaced at angular intervals of 18°. With the four-stroke cycle, the fact that one crank is 162° ahead of the other, permits of the explosions taking place alternately on each crank after an angular travel of 36°. Considered as two radials of ten cylinders each, we find that outside the primary the only unbalanced inertia-force harmonics are the 9th and 11th, 19th, 21st, etc., that is, the odd harmonics occasioned by the offset. As the primary inertia-force can be balanced by counterbalance masses on the crankshaft, the want of balance of the inertia forces is very small indeed.

C. *Explosion-Torque.*

§ 63. **Two Cranks at 180°.**—The explosion-torque of the engines discussed in Art. 62 will be the same as if they were cylinder in line types. The six-cylinder Anzani gives three twin units, in each of which the explosion-torque completes its cycle in one revolution of the crank. The ten-cylinder Anzani gives five sets, and the twenty-cylinder ten sets of such units. The balance of the explosion-torque of the six-cylinder Anzani is identical with the balance of the explosion-torque in the six-cylinder in line type of same size, offset, etc. The ten-cylinder Anzani has unbalanced fifth harmonics only. From Table IV., page 48, we have

$$E_5 = 3{\cdot}5 \sin 5\theta - 10{\cdot}6 \cos 5\theta$$
$$= 11{\cdot}1 \sin (5\theta + \epsilon) \text{ lbs. per twin set for pistons 1 sq. in. area.,}$$

the maximum value of which is 11·1 lbs. In other words, the maximum unbalanced explosion-torque on the crankshaft (and on the engine frame) is $11{\cdot}1 \times 5\pi R^2 r$ lbs.-ft. In the twenty-cylinder Anzani the largest unbalanced harmonic of the explosion-torque is the tenth. The series in Art. 31 has not been carried far enough to include this harmonic, it being assumed that the coefficients beyond the sixth are zero. It is known that the twenty-cylinder Anzani runs with great smoothness.

§ 64. **Explosion-Torque: One-Crank Radial Engine.**—In the four-stroke-cycle engine the camshaft rotates at half the speed of the crankshaft. The table below gives the crank angles and corresponding cam angles throughout a whole cycle of operations for one cylinder. The table also sets out the crank-effort corresponding to the different crank angles, the figures being taken from Table III., page 37, for a rod four cranks long. It must be remembered that in the latter table the crank efforts for angles between 0° and 180° are for an explosion stroke, and the crank efforts for angles between 180° and 360° are for a compression stroke. For the single cylinder, therefore, the crank-effort for the explosion stroke will be set opposite the crank angles 0° to 180°; while the crank-effort for the compression stroke will be set opposite the crank angles 540 to 720. While the crank turns from 180° to 540°, we have the exhaust and intake strokes, so that we set against these angles zero crank-effort. With respect to the camshaft angle x, it will be observed that the crank-effort has a period 2π. If E and x be plotted, we can determine the coefficients in the equation

$$\begin{aligned} E = a_o + a_1 \cos x + a_2 \cos 2x + \ . \ . \ . + a_{12} \cos 12x \\ + b_1 \sin x + \ . \ . \ . + b_{11} \sin 11x, \end{aligned}$$

so that it will pass through the points as plotted from the table. The equation is taken to give the actual crank effort for any angle. The coefficients are determined as in Art. 34, and are given in Table II. below.

TABLE I.

Crank angle	0	30	60	90	120	150	180 to 540	570	600	630	660	690	720
Cam angle x	0	15	30	45	60	75	90 to 270	285	300	315	330	345	360
Crank effort E	0	175	140	80	40	17	0	− ·3	− 3	− 11	− 28	− 42	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6 to y_{18}	y_{19}	y_{20}	y_{21}	y_{22}	y_{23}	—

TABLE II.

Cosine Coefficients.		Sine Coefficients.	
a_0 = 15·3			
a_1 = 24·8	a_7 = − 6·5	b_1 = 21·5	b_7 = 8·6
a_2 = 11·6	a_8 = − 6·6	b_2 = 32·5	b_8 = 5·4
a_3 = − ·2	a_9 = − 5·9	b_3 = 31·2	b_9 = 3·2
a_4 = − 5·8	a_{10} = − 5·4	b_4 = 23·4	b_{10} = 2·1
a_5 = − 6·6	a_{11} = − 5·6	b_5 = 16·4	b_{11} = 1·2
a_6 = − 6·2	a_{12} = − 2·9	b_6 = 11·9	

§ 65. **Balance of Explosion-Torque Harmonics.**—The crank effort due to one cylinder is given by the equation for E in Art. 64. For an engine having c cylinders, all harmonics are in balance, except their orders are multiples of c. When n is a multiple of c the unbalanced nth harmonic, per cylinder, is

$$E_n = a_n \cos nx + b_n \sin nx.$$

Thus the nine-cylinder Jupiter radial-engine has the 9th harmonic of the explosion-torque unbalanced. The magnitude of this is given in terms of crank-effort by

$$E_9 = 3{\cdot}2 \sin 9x - 5{\cdot}9 \cos 9x \text{ lbs. per cylinder.}$$

Writing this in the form $A_9 \sin (9x + \epsilon)$ we get $A_9{}^2 = 3{\cdot}2^2 + 5{\cdot}9^2$, or $A_9 = 6{\cdot}72$. The maximum value of E_9 is therefore 6·72 lbs. If the crank radius be 3·75 inches, this gives a torque of $6{\cdot}72 \times 3{\cdot}75 = 25{\cdot}2$ lbs.-inches. If the bore be 5·75 inches, we have the area of the nine pistons equal to 233·64 square inches, so that the maximum amount of explosion-torque fluctuation is $25{\cdot}2 \times 233{\cdot}64 \doteqdot 481$ lbs.-ft. If we wish to obtain the combined torque due to inertia and explosion, we must refer the former torque, which is $B_9 \sin 9\theta$, to the cam shaft angle x, when it becomes $B_9 \sin 18x$. The total torque, omitting harmonics of higher order is, therefore,

$$A_9 \sin (9x + \epsilon) + A_{18} \sin (18x + \epsilon') + B_9 \sin 18x$$

where $B_9 = mr^2\omega^2 t_9$.

The coefficient t_9 is certainly less than the coefficient t_6 given on page 21, so that the fluctuation in crankshaft torque for this engine is sensibly that due to explosion alone.

CHAPTER VI.

ROTARY ENGINE.

§ 66. **Piston Acceleration.**—When the cylinder makes an angle θ with the vertical, let the distance of the piston from the crankshaft centre A (Fig. 25) be x. We have the vector displacement of the piston with respect to origin at A, and the vertical line AC as standard direction, given by AB $= z$, where

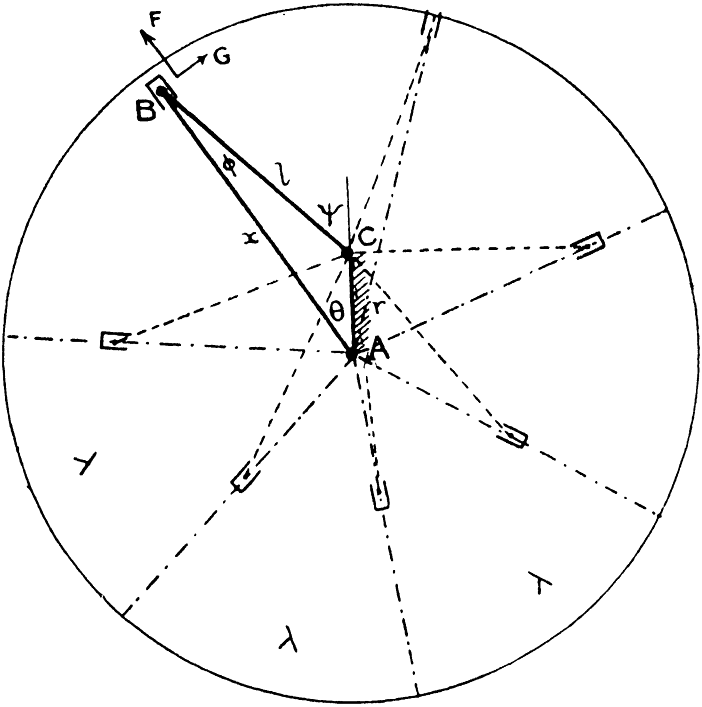

FIG. 25.—Rotating cylinder engine.

$$z = xe^{i\theta}.$$

The piston velocity is therefore

$$\dot{z} = \dot{x}e^{i\theta} + xie^{i\theta}\dot{\theta}.$$

Differentiating this, we get the piston acceleration

$$\ddot{z} = \ddot{x}e^{i\theta} + \dot{x}ie^{i\theta}\dot{\theta} + \dot{x}ie^{i\theta}\dot{\theta} + xi^2e^{i\theta}\dot{\theta}^2 + xie^{i\theta}\ddot{\theta}.$$

$$\therefore \ddot{z} = e^{i\theta}(\ddot{x} + 2i\dot{\theta}\dot{x} - x\dot{\theta}^2 + ix\ddot{\theta}).$$

Writing ω for $\dot{\theta}$, the angular velocity of the engine, and assuming ω constant, the last term above is zero, so that the piston acceleration is

$$\ddot{z} = e^{i\theta}\,(\ddot{x} + 2i\omega\dot{x} - x\omega^2)$$

From Art. 16 we have

$$x = r(\cos\theta + q_o + q_2\cos 2\theta + q_4\cos 4\theta + \ldots)$$
$$\dot{x} = -r\omega(\sin\theta + 2q_2\sin 2\theta + 4q_4\sin 4\theta + \ldots)$$
$$\ddot{x} = -r\omega^2(\cos\theta + 4q_2\cos 2\theta + 16q_4\cos 4\theta + \ldots).$$

Setting these values in the expression for $\ddot{z}$, we get

$$\ddot{z} = -r\omega^2e^{i\theta}\{q_o + 2\cos\theta + 5q_2\cos 2\theta + 17q_4\cos 4\theta + \ldots + 2i\sin\theta + 2i(2q_2\sin 2\theta + 4q_4\sin 4\theta + \ldots)\}.$$

§ 67. **Inertia-Resistance of Piston.**—Multiplying the acceleration $\ddot{z}$ by $-m$, we get the inertia resistance of the piston given by

$$F = mr\omega^2e^{i\theta}\{q_o + 2e^{i\theta} + 5q_2\cos 2\theta + 17q_4\cos 4\theta + \ldots + i(4q_2\sin 2\theta + 8q_4\sin 4\theta + \ldots)\}$$

Let $F_1 = mr\omega^2e^{i\theta}(q_o + 2e^{i\theta}) = mr\omega^2(q_oe^{i\theta} + 2e^{2i\theta})$. This part of F we shall call the fundamental inertia force.

§ 68. **Balance of Fundamental Inertia-Force.**—For an engine having c cylinders set at λ, the cylinders will make angles θ, $\theta + \lambda$, $\theta + 2\lambda$, . . . $\theta + \overline{c - 1}\lambda$ with the vertical, so that the total fundamental inertia-force becomes

Cylinder 0, .	$mr\omega^2\{q_0e^{i\theta} + 2e^{2i\theta}\}$
Cylinder 1, .	$mr\omega^2\{q_0e^{i(\theta + \lambda)} + 2e^{2i(\theta + \lambda)}\}$
Cylinder 2, .	$m\cdot\omega^2\{q_0e^{i(\theta + 2\lambda)} + 2e^{2i(\theta + 2\lambda)}\}$
. . . . .	. . . . . .
Cylinder $c - 1$, .	$mr\omega^2\{q_0e^{i(\theta + \overline{c - 1}\lambda)} + 2e^{2i(\theta + \overline{c - 1}\lambda)}\}.$

Hence adding

$$F_1 = mr\omega^2\{q_0e^{i\theta}(1 + e^{i\lambda} + e^{2i\lambda} + \ldots + e^{(c-1)i\lambda}) + 2e^{2i\theta}(1 + e^{2i\lambda} + e^{4i\lambda} + \ldots + e^{2(c-1)i\lambda})\}$$

From Art. 18 we know that the series in circumflex brackets are both zero unless $\lambda = 0°$ when neither series is zero, or when $\lambda = 180°$ when the second series is not zero. Hence the fundamental inertia-force of a multicylinder rotary engine is in balance if the number of cylinders be greater than two. With one cylinder corresponding to $\lambda = 0$, the fundamental inertia-force is

$$F_1 = mr\omega^2 q_0 e^{i\theta} + 2mr\omega^2 e^{2i\theta}.$$

The first term here is identical with the inertia-resistance (centrifugal force) of a mass m rotating with the engine at a distance $q_0 r$ from A (Fig. 25). The second term is identical with the inertia-resistance of a mass $\frac{1}{2}m$ rotating at twice the engine speed at a distance r from A. In a two-cylinder engine, corresponding to $\lambda = 180°$, the fundamental inertia-force is

$$F_1 = 4mr\omega^2 e^{2i\theta},$$

which is equivalent to the centrifugal force of a mass m rotating at a radius r with twice the engine speed.

§ 69. **Balance of Higher Harmonics.**—The terms involving q_n for cylinder making an angle θ with the vertical, are from Art. 66:

$$\begin{aligned} & mr\omega^2 e^{i\theta}\{(n^2+1)q_n \cos n\theta + i2nq_n \sin n\theta\} \\ &= mr\omega^2 e^{i\theta}\{\tfrac{1}{2}(n^2+1)q_n(e^{ni\theta} + e^{-ni\theta}) + nq_n(e^{ni\theta} - e^{-ni\theta})\} \\ &= \tfrac{1}{2}q_n mr\omega^2\{(n+1)^2 e^{(n+1)i\theta} + (n-1)^2 e^{-(n-1)i\theta}\} \\ &= Ke^{(n+1)i\theta} + Le^{-(n-1)i\theta}, \end{aligned}$$

where $K = \frac{1}{2}q_n mr\omega^2(n+1)^2$ and $L = \frac{1}{2}q_n mr\omega^2(n-1)^2$.

The total value of these terms for the engine is therefore:

Cylinder 0,	$Ke^{(n+1)i\theta} + Le^{-(n-1)i\theta}$
Cylinder 1,	$Ke^{(n+1)i(\theta+\lambda)} + Le^{-(n-1)i(\theta+\lambda)}$
Cylinder 2,	$Ke^{(n+1)i(\theta+2\lambda)} + Le^{-(n-1)i(\theta+2\lambda)}$
. . . .	. . . .
Cylinder $c-1$,	$Ke^{(n+1)i(\theta+\overline{c-1}\lambda)} + Le^{-(n-1)i(\theta+\overline{c-1}\lambda)}$.

which on summing gives

$$\begin{aligned} & Ke^{(n+1)i\theta}\{1 + e^{(n+1)i\lambda} + e^{(n+1)2i\lambda} + \ldots + e^{(n+1)(c-1)i\lambda}\} \\ & + Le^{-(n-1)i\theta}\{1 + e^{-(n-1)i\lambda} + e^{-(n-1)2i\lambda} \\ & \qquad + \ldots + e^{-(n-1)(c-1)i\lambda}\} \end{aligned}$$

From Art. 18, this sum is zero unless $n+1$ or $n-1$ be a multiple of c. The unbalanced harmonics are therefore those whose coefficients involve $q_{kc\pm1}$. The 7-cylinder Gnome has unbalanced harmonics whose coefficients involve q_6, q_8, q_{13}, q_{15}, etc. The terms involving q_6, q_8 give an unbalanced 7th harmonic which is $7(\mathrm{K}e^{7i\theta} + \mathrm{L}e^{-7i\theta})$. The unbalanced harmonic $7\mathrm{K}e^{7i\theta}$, is equivalent to the inertia resistance of a mass rotating at seven times the engine speed. The unbalanced harmonic $7\mathrm{L}e^{-7i\theta}$, is equivalent to the inertia resistance of a mass rotating in the opposite sense to the engine, and at seven times its speed.

§ 70. **Inertia-Torque Due to Piston.**—At the instant shown in Fig. 25 the inertia forces on the piston B are (1) a force $\mathrm{F} = mx\omega^2 - m\ddot{x}$ along AB; and (2) a force $\mathrm{G} = 2m\dot{x}\omega$ at right angles to AB. This follows from the expression for the piston acceleration,

$$\ddot{z} = e^{i\theta}(\ddot{x} + 2i\omega\dot{x} - x\omega^2).$$

The middle term is clearly an acceleration at right angles to AB, while the other two terms are accelerations along AB. It may be shown, as in Art. 21, that the force F causes an equal force, F, on the bearing at A, together with a couple $\mathrm{F}\dot{x}/\dot{\theta}$ on the cylinder, and a couple $-\mathrm{F}\dot{x}/\dot{\theta}$ on the engine-frame AC. The force G acting at right angles to AB causes an equal force, G, on the bearing at A, together with a couple $-\mathrm{G}x$ on the cylinder. The balance of the forces at the propeller-shaft-bearing A have been investigated in Arts. 67, 68, 69. We have now to investigate the balance of the couples on the engine-frame and on the cylinder. The latter couple, which we shall call the propeller-shaft inertia-torque, is

$$\begin{aligned} \mathrm{P} = \mathrm{F}\dot{x}/\dot{\theta} - Gx &= mx\dot{x}\omega - mx\dot{x}/\omega - 2mx\dot{x}\omega \\ &= -m\ddot{x}\dot{x}/\omega - mx\dot{x}\omega. \end{aligned}$$

The engine-frame torque N is $m\ddot{x}\dot{x}/\omega - mx\dot{x}\omega$. This shows that the inertia-resistance of pistons of a rotating-cylinder engine gives (1) a propeller-shaft torque which is the reverse of the crankshaft inertia-torque in the fixed-cylinder engine, and, (2) a torque $-mx\dot{x}\omega$. The engine-frame torque due to piston inertia of rotary is also reverse of the engine-frame torque of fixed-cylinder, together with a torque $-mx\dot{x}\omega$. Referring to Example 19, Appendix, this latter torque can be written

$$\begin{aligned} -mx\dot{x}\omega = \tfrac{1}{2}mr^2\omega^2\{(2q_0 + q_2)\sin\theta + 2\sin 2\theta + 3(q_2 + q_4)\sin 3\theta \\ + 5(q_4 + q_6)\sin 5\theta + \ldots\} \end{aligned}$$

Those orders which are integer multiples of the number of cylinders are the only harmonics of this torque unbalanced. The complete torque can be written down by reference to Art. 23.

§ 71. **Inertia-Torque Due to Rods.**—Replacing the rod by an equivalent dynamical system consisting of a mass m_2 at the gudgeon pin, and a mass m_1 at the big end, together with a rim of mass Δm and moment of inertia I′, as defined in Art. 25, we can treat m_2 as part of the piston mass ; m_1 at the big end being at rest, has no dynamic effect. The effect of I′ is to give (1) a propeller-shaft torque which is the same as the engine-frame torque in a fixed cylinder, and (2) an engine-frame torque which is the same as the crank torque in a fixed cylinder engine. To arrive at this result directly, we have the couple T on the rod given by $T = I'\ddot{\psi}$. Also, if P be the propeller shaft torque due to T, we have

$$\begin{aligned} P \,.\, \delta\theta &= T\delta\psi \\ \therefore\ P = T\dot{\psi}/\dot{\theta} &= T(\dot{\theta} + \dot{\phi})/\dot{\theta} \\ &= I'\ddot{\phi}(1 + \dot{\phi}/\dot{\theta}). \end{aligned}$$

The engine-frame torque is $T - P = - I'\ddot{\phi}\dot{\phi}/\dot{\theta}$. We may refer to Arts. 59 and 27 for torque due to I′.

§ 72. **Explosion-Torque.**—The explosion-torque in the rotating-cylinder engine is the same as in the corresponding radial engine, given in Art. 65.

§ 73. **Kinetic Energy of Complete Mechanism.**—The kinetic energy due to reciprocating piston mass is $\frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$. The rod is most conveniently dealt with as consisting of detached masses at crank and gudgeon pins, and a fictitious moment of inertia I′. The mass at the gudgeon pin may be added to the mass of the piston. The mass at the crank-pin will have no kinetic energy. The kinetic energy due to I′ is

$$\tfrac{1}{2}I'\dot{\psi}^2 = \tfrac{1}{2}I'(\dot{\theta} + \dot{\phi})^2.$$

Exercise.—Discuss the balance of the nine-cylinder Le Rhone engine.

§ 74. **Rotating Cylinders and Rotating Crank.**—In the two-throw crank, six-cylinder E.J.C. engine, both cylinders and crankshaft rotate. The relative speed is about 2000 r.p.m. The absolute speeds of cylinders and crank depend on the resistances experienced by the two propellers to which they are attached. Roughly, the crankshaft with its propeller makes 1200 r.p.m., while the crankcase and cylinders, with its propeller, make 800 r.p.m. in the opposite direction. As in Art. 66, we get the piston acceleration given by

$$\ddot{z} = e^{i\gamma}(\ddot{x} + 2i\Omega\dot{x} - x\Omega^2)$$

where Ω is the angular velocity of the cylinders. Substituting for x, $\dot{x}$, $\ddot{x}$ from Art. 16, we get, writing ω for $\dot{\theta}$.

$$\begin{aligned}\ddot{z} = &- re^{i\gamma}\{\Omega^2 q_0 + (\omega^2 + \Omega^2)\cos\theta + 2i\Omega\omega\sin\theta \\ &+ (\Omega^2 + 4\omega^2)q_2\cos 2\theta + \ldots + 4iq_2\Omega\omega\sin 2\theta + \ldots\}\end{aligned}$$

$$\therefore\ \mathrm{F} = mre^{i\gamma}\{\Omega^2 q_0 + \text{etc.}\}.\quad \text{Now } \theta = \gamma - \alpha.$$

$$\begin{aligned}\therefore\ \mathrm{F}e^{-i\alpha} &= mre^{i\theta}\{\Omega^2 q_0 + \tfrac{1}{2}(\omega^2 + \Omega^2)(e^{i\theta} + e^{-i\theta}) + \Omega\omega(e^{i\theta} - e^{-i\theta}) \\ &\qquad + \ldots\} \\ &= mr\{\Omega^2 q_0 e^{i\theta} + \tfrac{1}{2}(\omega^2 + \Omega^2 - 2\omega\Omega) + \tfrac{1}{2}(\omega^2 + \Omega^2 + 2\omega\Omega)e^{2i\theta} \\ &\qquad + e^{i\theta}(k_2\cos 2\theta + \ldots + il_2\sin 2\theta + \ldots)\}.\end{aligned}$$

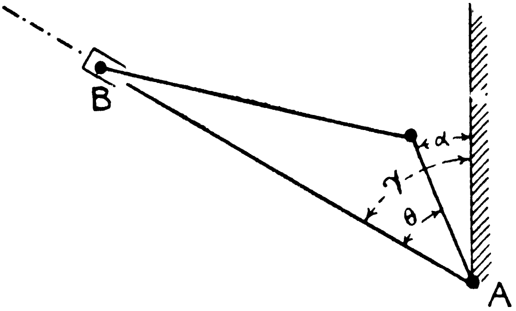

FIG. 26.—Cylinders and crank rotating.

For a complete engine in which the cylinders make angles θ, $\theta + \lambda$, $\theta + 2\lambda$, . . . , with the crank, we get for the first three terms in bracketed portion above :

Cylinder 0, $\Omega^2 q_0 e^{i\theta} + \tfrac{1}{2}(\omega - \Omega)^2 + \tfrac{1}{2}(\omega + \Omega)^2 e^{2i\theta}$
Cylinder 1, $\Omega^2 q_0 e^{i(\theta + \lambda)} + \tfrac{1}{2}(\omega - \Omega)^2 + \tfrac{1}{2}(\omega + \Omega)^2 e^{2i(\theta + \lambda)}$
Cylinder 2, $\Omega^2 q_0 e^{i(\theta + 2\lambda)} + \tfrac{1}{2}(\omega - \Omega)^2 + \tfrac{1}{2}(\omega + \Omega)^2 e^{2i(\theta + 2\lambda)}$

. . . .

Cylinder $c - 1$, $\Omega^2 q_0 e^{i(\theta + \overline{c-1}\lambda)} + \tfrac{1}{2}(\omega - \Omega)^2$
$+ \tfrac{1}{2}(\omega + \Omega)^2 e^{2i(\theta + \overline{c-1}\lambda)}$

$$\begin{aligned}&= \Omega^2 q_0 e^{i\theta}(1 + e^{i\lambda} + e^{2i\lambda} + \ldots + e^{(c-1)i\lambda}) \\ &\quad + \tfrac{1}{2}c(\omega - \Omega)^2 \\ &\quad + \tfrac{1}{2}(\omega + \Omega)^2 e^{2i\theta}(1 + e^{2i\lambda} + e^{4i\lambda} + \ldots + e^{2(c-1)i\lambda}).\end{aligned}$$

The two series in this expression vanish for engines of more than two cylinders, so that we are left with but the term $\frac{1}{2}c(\omega - \Omega)^2$. We may write this portion of F as

$$F_1e^{-i\alpha} = \tfrac{1}{2}mrc(\omega - \Omega)^2,$$

or

$$F_1 = \tfrac{1}{2}mrc(\omega - \Omega)^2e^{i\alpha}.$$

If ω_1 be the angular velocity of the crank we have $\omega = \Omega - \omega_1$, so that

$$F_1 = \tfrac{1}{2}mrc\omega_1{}^2e^{i\alpha}.$$

This is the same force as exists in a radial engine of corresponding type examined in Art. 51. With the two-throw crank of the E.J.C. the F_1 force of one set of three cylinders balances that of the other set.

The balance of the remaining harmonics of $F . e^{-i\alpha}$, namely, $e^{i\theta}(k_n \cos n\theta + il_n \sin n\theta)$ is investigated as in Art. 69. We obtain the same result as there given.

§ 75. **Zeitlin Nine-cylinder Rotary-Engine.**—The mechanism of one cylinder of this engine is shown in Fig. 27. The big end D of the connecting rod DB works on an eccentric whose centre D rotates about the fixed centre C with half the engine speed. The figures 1, 2, 3, 4 show corresponding positions of the centre D of the eccentric and the piston B. From 1 to 2 gives the induction stroke (226 mm.) ; 2 to 3 the compression stroke (203·5 mm.) ; 3 to 4 the combustion stroke (181 mm.) ; and from 4 to 1 the exhaust stroke (203·5 mm.). The figures correspond to a crank radius r = AC of 101·75 mm., and an eccentric radius q = CD of 11·125 $\sqrt{2}$ mm. There are nine eccentric discs to take the nine rods. The eccentric radius CD lies along CE when the cylinder AB is vertical. Angle ACE is $\frac{3\pi}{4}$.

From the figure we get

$$x = r\cos 2\omega t + q\cos\left(\omega t + \tfrac{\pi}{4}\right) + l\sqrt{1 - \sin^2\phi}$$
$$l\sin\phi = r\sin 2\omega t + q\sin\left(\omega t + \tfrac{\pi}{4}\right).$$

Expanding the radical by the Binomial Theorem, and substituting for sin ϕ from the second equation, we obtain by the method of Art. 38, or otherwise, a series for x of the form :

$$x = a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \dots$$
$$+ b_1\sin\omega t + b_2\sin 2\omega t + \dots$$

The acceleration of B is from Art. 66 :

$$\ddot{z} = e^{i2\omega t}(\ddot{x} + 4i\dot{x}\omega - 4x\omega^2).$$

Substituting for x, $\dot{x}$, $\ddot{x}$ derived from the series for x, and recollecting that $e^{i\omega t} = \cos \omega t + i \sin \omega t$, we get

$$\begin{aligned}\ddot{z} = -\omega^2 e^{2i\omega t}\{4a_0 &+ 4(a_1 - ib_1)e^{i\omega t} + 8(a_2 - ib_2)e^{2i\omega t} \\ &+ 12(a_3 - ib_3)e^{3i\omega t} + \dots \\ &+ a_1 \cos \omega t + a_3 \cos 3\omega t + 4a_4 \cos 4\omega t + 9a_5 \cos 5\omega t + \dots \\ &+ b_1 \sin \omega t + b_3 \sin 3\omega t + 4b_4 \sin 4\omega t + 9b_5 \sin 5\omega t + \dots\}\end{aligned}$$

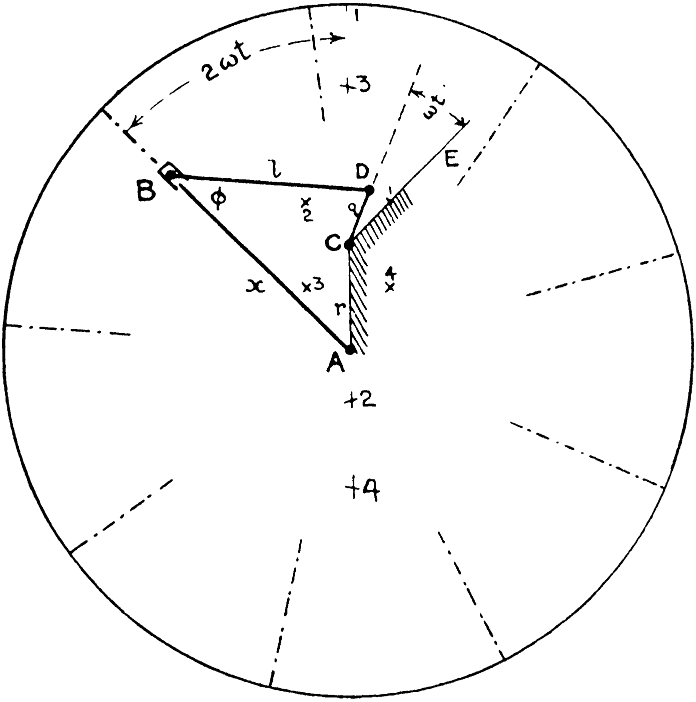

FIG. 27.—Zeitlin nine-cylinder rotary engine with long induction and short combustion stroke.

Multiplying by $-m$, we get the inertia resistance of the piston:

$$\begin{aligned}F = c_0 e^{2i\omega t} + c_1 e^{3i\omega t} + c_2 e^{4i\omega t} + \dots &+ e^{2i\omega t}\{d_1 \cos(\omega t - \epsilon) \\ &+ d_3 \cos(3\omega t - \epsilon_3) + d_4 \cos(4\omega t - \epsilon_4) + \dots\}\end{aligned}$$

where $c_0 = 4m\omega^2 a_0$, $c_1 = 4m\omega^2(a_1 - ib_1)$, etc.

In a multicylinder, the eccentric radii CD will make angles ωt, $\omega t + \lambda$, $\omega t + 2\lambda$, . . . with CE The terms of F which involve the subscript n are therefore, for the complete engine

$$
\begin{aligned}
\mathrm{F}_n = {} & c_n\{e^{(n+2)i\omega t} + e^{(n+2)i(\omega t+\lambda)} + e^{(n+2)i(\omega t+2\lambda)} + \ldots\} \\
& + d_n e^{-i\epsilon_n}\{e^{(n+2)i\omega t} + e^{(n+2)i(\omega t+\lambda)} + \ldots + e^{-(n-2)i\omega t} \\
& \qquad + e^{-(n-2)(i)(\omega t+\lambda)} + \ldots\} \\
= {} & c_n e^{(n+2)i\omega t}\{\mathrm{I} + e^{i\lambda(n+2)} + e^{2i\lambda(n+2)} + \ldots\} \\
& + d_n e^{-i\epsilon_n} e^{(n+2)i\omega t}\{\mathrm{I} + e^{i\lambda(n+2)} + e^{2i\lambda(n+2)} + \ldots\} \\
& + d_n e^{-i\epsilon_n} e^{-(n-2)i\omega t}\{\mathrm{I} + e^{-i\lambda(n-2)} + e^{-2i\lambda(n-2)} + \ldots\}
\end{aligned}
$$

Hence F_n is zero except when $n \pm 2$ is a multiple of the number of cylinders. This result is reconciled with that of Art. 69, if it be noted that the coefficients with suffix n in the series for x as referred to eccentric angle in this Art., correspond to coefficients with suffix $2n$ when referred to angles made by cylinders (crank angles).

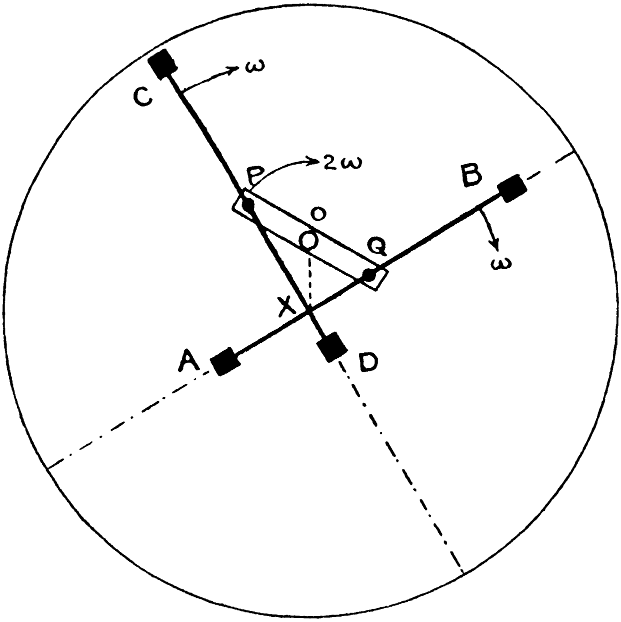

FIG. 28.—Burlat rotating cylinder engine.

§ 76. **Burlat Rotary Engine.**—In the Burlat rotary engine (Fig. 28) the crank PQ rotates about the crankshaft centre O with angular speed 2ω. The rods AB, CD rotate with speed ω about the crank-pins Q, P. Clearly, if the rods originally cross at right angles, they will always remain at right angles, since their angular speeds are the same. A semi-circle on PQ will therefore pass through X, so that OX is a fixed length. Further, in all positions, angle DXO equals angle QPC, for OPX = OXP; therefore velocity of CD relative to OX is equal to the velocity of PQ relative to CD, which is ω. Since the actual velocity of CD is equal to its velocity relative to OX, OX must be at rest, so that X is a fixed point for all positions of the mechanism. The rods AB, CD have pistons bolted to their ends, which work in cylinders which rotate about X with angular velocity ω. The piston stroke is 4OQ.

The rod AB is dynamically equivalent to a mass m at its centre of gravity Q, and a rim of moment of inertia I where I is the moment of inertia of AB about Q. Since the angular velocity of AB is uniform, we have no couples due to its moment of inertia. The masses m, m at the crank-pins balance each other, so that all inertia effects are in perfect balance. There will be a disturbance, however, due to explosion torque on engine frame, as with the other types already dealt with.

CHAPTER VII.

OSCILLATING-CYLINDER ENGINE.

§ 77. **Equivalent Dynamical System.**—In the mechanism of the oscillating-cylinder (Fig. 29), let the distance between the centre of the crankshaft B and the axis of oscillation of the cylinder A (the crankshaft trunnion distance) be denoted by c; let a be the crank radius, and let $a/c = \rho$. In Fig. 29, θ is the angle made by the crank with the line of centres, ϕ the angular displacement of the cylinder from the line of centres, and ψ the angle between the crank and piston (rod). The cylinder may be resolved into an equivalent dynamical system of two detached masses; one, m_1, at the trunnions A, and the other, m_2, at Q, a distance k from the trunnions. To have mass systems dynamically equivalent, it is necessary and sufficient that the systems have the same mass, moment of mass, and moment of inertia about any chosen axis. Hence, if m be the mass of the cylinder and $\bar{x}$ the distance of the centre of gravity of the cylinder from the trunnions, we have, on taking moments of mass about the trunnions,

$$m_2 k = m\bar{x}.$$

Taking moments of inertia about the trunnions, we have

$$m_2 k^2 = \mathrm{I}$$

where I is the moment of inertia of the cylinder about the trunnions. These two equations enable us to determine both m_2 and k, which we may therefore assume to be known for any particular cylinder. The case in which the trunnions pass through the centre of gravity of the cylinder is dealt with by writing $m_2 k^2 = \mathrm{I}$ and $m_2 k = 0$ in any expression in which these quantities occur. The dynamic effect of the cylinder due to its oscillation is the same as that due to m_2 since the mass m_1 remains at rest.

The vector displacement of m_2 is AQ. Denoting this by z_2, we have, taking BA as our standard direction,

$$z_2 = ke^{-i\phi}.$$

The velocity of m_2 is therefore

$$\dot{z}_2 = -ike^{-i\phi}.\,\phi.$$

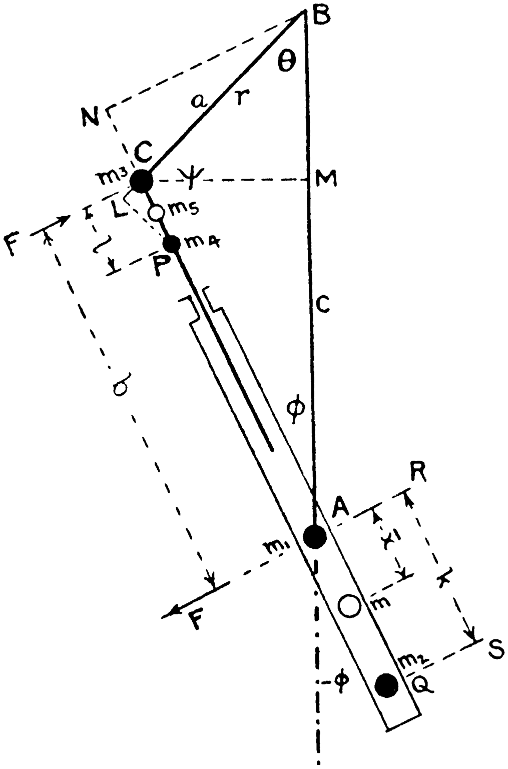

FIG. 29.—Oscillating cylinder.

The acceleration of m_2 is

$$\ddot{z}_2 = -e^{-i\phi}.\,k\dot{\phi}^2 - ie^{-i\phi}k\ddot{\phi}.$$

§ 78. **Inertia Effects of Cylinder.**—Reversing the signs in the expression for $\ddot{z}_2$ and multiplying by m_2, we get the inertia resistance due to m_2 to be given by

$$e^{-i\phi}.\,m_2k\dot{\phi}^2 + ie^{-i\phi}.\,m_2k\ddot{\phi} \quad . \quad . \quad . \quad (1)$$

The first term in this expression, being a force through m_2 along the cylinder, gives a load

$$e^{-i\phi} . m_2k\dot{\phi}^2 \quad . \quad . \quad . \quad . \quad (\text{I.})$$

on the trunnion-bearing.

The other term is a force through m_2 at right angles to the cylinder, that is, along SQ.

§ 79. **Effects of Forces on Cylinder at Right Angles to its Axis.**—The above force along SQ (Fig. 29) causes forces on the crank-pin and on the trunnion-bearing which we can calculate by considering the rod and cylinder as forming a single rigid beam CAQ, loaded at right angles to its length at Q, and supported at C and A by reactions also at right angles to CA. The reactions at C and A are perpendicular to the common axis CA of the cylinder and the rod, because the force along SQ, at Q, transmits its effect to the rod by pressures at the piston-rings and stuffing-box where the rod and cylinder are in contact, and the pressures at these contact points, acting as they do (in the absence of friction which is assumed) at right angles to CA, must give a parallel pressure on the crank-pin. Now a load W at Q and along SQ on the beam CAQ will cause a parallel load W $(k + b)/b$ on the support at A, and a load Wk/b in the opposite direction on the support at C. These results follow at once by taking moments about the supports. The force $ie^{-i\phi}. m_2k\ddot{\phi}$ through m_2, therefore, causes a load

$$ie^{-i\phi} . m_2k\ddot{\phi}(k + b)/b \quad . \quad . \quad . \quad (\text{II.})$$

on the trunnion-bearing at A, and a load

$$- ie^{-i\phi}m_2k\ddot{\phi}k/b \quad . \quad . \quad . \quad (\text{III.})$$

on the crank-pin at C.

§ 80. **Inertia Effects of Rod and Piston.**—The rod and piston may likewise be replaced by an equivalent dynamical system of two masses, one of which, m_3, it will be convenient to place at the crank-pin. Let the other mass m_4 be at a distance l from the crank-pin. If m_5 be the mass of the piston and rod, and $\bar{y}$ the distance of its centre of gravity from the crank-pin, the conditions for dynamical equivalence require

$$\begin{aligned} m_3 &= m_5 - m_4 \\ m_4l &= m_5\bar{y} \\ m_4l^2 &= \text{I}_1 \end{aligned}$$

where I_1 is the moment of inertia of the piston and rod about the crank-pin end. We can therefore determine m_3, m_4, and l from the above equations. These quantities will be now assumed known for the engine we are considering.

The vector displacement of the mass m_3 at the crank-pin is BC. Denoting this vector by z_3 we have

$$z_3 = re^{i\theta}\,;\ \therefore\ \dot{z}_3 = ire^{i\theta}\dot{\theta}\,;\ \therefore\ \ddot{z}_3 = -\,re^{i\theta}\,.\,\dot{\theta}^2.$$

The inertia resistance of m_3 is therefore

$$e^{i\theta}\,.\,m_3r\dot{\theta}^2 \qquad . \quad . \quad . \quad . \quad \text{(IV.)}$$

This force causes an equal load on the crank-pin.

Denoting the vector displacement of m_4 at P by z_4 we have

$$z_4 = \text{BP} = \text{BC} + \text{CP}.$$
$$\therefore\ z_4 = re^{i\theta} + le^{-i\phi}.$$

Hence the velocity of m_4 is

$$\dot{z}_4 = ire^{i\theta}\,.\,\dot{\theta} - ile^{-i\phi}\,.\,\dot{\phi},$$

and the acceleration of m_4 is

$$\begin{aligned}\ddot{z}_4 &= i^2re^{i\theta}\,.\,\dot{\theta}^2 + i^2le^{-i\phi}\,.\,\dot{\phi}^2 - ile^{-i\phi}\ddot{\phi}\\ &= -\,e^{i\theta}\,.\,r\dot{\theta}^2 - e^{-i\phi}l\dot{\phi}^2 - ile^{-i\phi}\ddot{\phi}.\end{aligned}$$

The inertia resistance of m_4 is therefore

$$e^{i\theta}m_4r\dot{\theta}^2 + e^{-i\phi}\,.\,m_4l\dot{\phi}^2 + ie^{-i\phi}m_4l\ddot{\phi} \quad . \quad . \quad . \quad (2)$$

The first term in (2) is a force through m_4 parallel to the crank. This force can be resolved into a component

$$e^{-i\phi}\,.\,m_4r\dot{\theta}^2\,.\,\text{NC/CB} \quad . \quad . \quad . \quad (3)$$

along the rod, and a component

$$ie^{-i\phi}\,.\,m_4r\dot{\theta}^2\,.\,\text{NB/CB} \quad . \quad . \quad . \quad (4)$$

at right angles to the rod. The component (3) causes an equal load

$$e^{-i\phi}\,.\,m_4\dot{\theta}^2\,.\,\text{NC} \quad . \quad . \quad . \quad \text{(V.)}$$

on the crank-pin. Now it may be shown, as in Art. 79, that a load W acting at right angles to the rod through m_4 will cause a load $W(b - l)/b$ on the crank-pin, and a load Wl/b on the trunnion-bearing. The effect of the component (4) above is therefore to cause a load

$$ie^{-i\phi}\ m_4\dot{\theta}^2\text{NB}(b - l)/b \quad . \quad . \quad \text{(VI.)}$$

on the crank-pin, and a load

$$ie^{-i\phi} . m_4\dot{\theta}^2\mathrm{NB} . l/b \quad . \quad . \quad . \quad \text{(VII.)}$$

on the trunnion-bearing.

The third term in (2) is a force through m_4 at right angles to the cylinder axis. It will be evident from what has been said above, that this force causes a load

$$ie^{-i\phi} . m_4l\ddot{\phi}(b - l)/b \quad . \quad . \quad . \quad \text{(VIII.)}$$

on the crank-pin, and a load

$$ie^{-i\phi} . m_4l\ddot{\phi}l/b \quad . \quad . \quad . \quad \text{(IX.)}$$

on the trunnion-bearing.

Finally, the second term in (2) is a force along the rod. It is, in fact, the so-called centrifugal force due to the angular velocity of m_4 about C. This force evidently causes a load

$$e^{-i\phi} . m_4l\dot{\phi}^2 \quad . \quad . \quad . \quad . \quad \text{(X.)}$$

on the crank-pin.

§ 81. **Summary of Results.**—Due to inertia, we therefore have acting on the crank-pin the loads IV., V., X., VI., VIII., and III., the resultant of which is the vector sum

$$\begin{aligned} &e^{i\theta} . m_3r\dot{\theta}^2 + e^{-i\phi}[m_4(\dot{\theta}^2 . \mathrm{NC} + l\dot{\phi}^2)] \\ &\qquad + ie^{-i\phi}[m_4(\dot{\theta}^2 . \mathrm{NB} + l\dot{\phi})(b - l)/b - m_2k^2\ddot{\phi}/b] \quad . \quad (5) \end{aligned}$$

Acting on the trunnion-bearing we have the loads I., II., VII., and IX., the resultant of which is the vector sum

$$e^{-i\phi}m_2k\dot{\phi}^2 + ie^{-i\phi}[m_2k\ddot{\phi}(k + b)/b + m_4l(\dot{\theta}^2 . \mathrm{NB} + l\ddot{\phi})/b] \quad (6)$$

§ 82. **Effect of Inertia-Loads on Crank-pin.**—The effect of the crank-pin inertia-loads is to give a set of equal loads on the crankshaft-bearing, together with a torque on the crankshaft. To obtain the crankshaft inertia-torque we take the sum of the moments about the crankshaft centre of each inertia-load on the crank-pin.

§ 83. **Replacement of Trunnion-Loads by Equivalent Force-System.**—The effect of the trunnion-bearing inertia-loads in causing vibration of the engine-frame is the same as the combined effects of an equal set of loads acting through the crankshaft-bearing and a couple on the engine-frame, the moment of the couple being the sum of the moments about the crankshaft centre of the inertia-loads on the trunnion-bearing By this replacement, the complete dynamic effect of the engine motion is thus represented by

(1) A force on the engine-frame acting through the centre of the crankshaft.
(2) A couple on the engine-frame.
(3) A couple on the crankshaft.

§ 84. **The Inertia-Force on Engine-Frame through Crankshaft Centre.**—From Arts. 82 and 83 we see that the inertia-force on the engine-frame acting through the crankshaft centre is the sum of the loads acting on both crank-pin and trunnion-bearing, that is, the loads I. to X. set out in Art. 81. Summing these loads, we get the inertia-force on the engine-frame to be

$$\begin{aligned} \mathrm{F} = e^{i\theta}\, m_3 r\dot{\theta}^2 &+ e^{-i\phi}(\mathrm{NC} + i\mathrm{NB})m_4\dot{\theta}^2 \\ &+ e^{-i\phi}(m_2k + m_4l)\dot{\phi}^2 + ie^{-i\phi}(m_2k + m_4l)\ddot{\phi}. \end{aligned}$$

Now since the vector $e^{-i\phi}(\mathrm{NC} + i\mathrm{NB})$ is equal to

$$e^{-i\phi} \,.\, r[\cos(\theta + \phi) + i \sin(\theta + \phi)] = re^{-i\phi} \,.\, e^{i(\theta + \phi)} = re^{i\theta},$$

the first two terms in the expression for F become

$$e^{i\theta}(m_3 + m_4)r\dot{\theta}^2,$$

which plainly is equivalent to the dynamic effect of a mass $m_3 + m_4 = m_5$ at, and rotating with, the crank-pin. In a multicylinder engine with cranks symmetrically disposed, this portion of F is in collective balance, and in any engine its balance can be secured by suitable counterweights on the crankshaft.

The remaining two terms in the expression for F can be written

$$\begin{aligned} &(\cos\phi - i\sin\phi)(m_2k + m_4l)\dot{\phi}^2 + (i\cos\phi + \sin\phi)(m^2k + m_4l)\ddot{\phi} \\ &= (m_2k + m_4l)(\cos\phi\dot{\phi}^2 + \sin\phi\ddot{\phi}) + i(m_2k + m_4l)(\cos\phi\ddot{\phi} - \sin\phi\dot{\phi}^2). \end{aligned}$$

That is, we have a vertically downward load

$$\mathrm{V} = (m_2k + m_4l)(\dot{\phi}^2\cos\phi + \ddot{\phi}\sin\phi)$$

and a horizontal load

$$\mathrm{H} = (m_2k + m_4l)(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi).$$

Referring to Appendix (Examples 11, 10), we have

$$\begin{aligned} \mathrm{V} &= -\omega^2(m_2k + m_4l)(f_1\cos\theta + 4f_2\cos 2\theta + 9f_3\cos 3\theta + \ldots) \\ \mathrm{H} &= \omega^2(m_2k + m_4l)(a_1\sin\theta + 8a_2\sin 2\theta + 27a_3\sin 3\theta + \ldots). \end{aligned}$$

In a multicylinder engine we have balance of all harmonics except those which are multiples of the number of cylinders. A four-cylinder engine with cranks at 180° has an unbalanced secondary :

$$-\ \omega^2(m_2k + m_4l)\cdot0588 \cos 2\theta \text{ per cylinder vertically and}$$
$$-\ \omega^2(m_2k + m_4l)\cdot1232 \sin 2\theta \text{ per cylinder horizontally.}$$

An engine with cranks at 120° has an unbalanced third harmonic :

$$\cdot0333\omega^2(m_2k + m_4l) \cos 3\theta \text{ per cylinder vertically}$$
$$-\ \cdot0513\omega^2(m_2k + m_4l) \sin 3\theta \text{ per cylinder horizontally.}$$

§ 85. **The Inertia-Torque on the Engine-Frame.** — As stated in Art. 83, the moment of the couple on the engine-frame is the sum of the moments about the crankshaft centre of the loads on the trunnion-bearing. These loads are set out in (6), Art. 81. In calculating the moment about B of forces through A (Fig. 29), we note that forces along the cylinder have the lever-arm NB, while forces acting at right angles to the cylinder have the lever-arm AN. The moment of (6) about the crankshaft centre is therefore

$$-\ m_2k\dot{\phi}^2 \,.\, \mathrm{NB} + [m_2k\ddot{\phi}(k + b)/b + m_4l(\dot{\theta}^2 \,.\, \mathrm{NB} + l\ddot{\phi})/b]\mathrm{AN}.$$

Now $\mathrm{NB} = c \sin \phi$ and $\mathrm{AN} = c \cos \phi$. Further, since the velocity of C normal to AC is $b\dot{\phi}$, and is also $a\dot{\theta} \cos \psi$ or $\mathrm{CN} \,.\, \dot{\theta}$, we have $b\dot{\phi} = \mathrm{CN} \,.\, \dot{\theta}$. As $\mathrm{CN} = c \cos \phi - b$, we get $c \cos \phi = b + \mathrm{CN} = b(1 + \dot{\phi}/\dot{\theta})$. Making use of these relations, the above expression for the engine-frame torque reduces to

$$m_2kc(\ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi) + (m_2k^2 + m_4l^2)(1 + \dot{\phi}/\dot{\theta})\ddot{\phi}$$
$$+\ m_4l^2\dot{\theta}^2 \sin \phi \,.\, (1 + \dot{\phi}/\dot{\theta}).$$

Referring to the Appendix, Ex. 10, 8, 9, 11, it will be seen that we can write the above expression in a series of the form

$$\dot{\theta}^2(\mathrm{A}_1 \sin \theta + \mathrm{A}_2 \sin 2\theta + \mathrm{A}_3 \sin 3\theta + \ldots + \mathrm{A}_n \sin n\theta + \ldots).$$

In a multicylinder engine all harmonics of the engine-frame inertia-torque are in balance except the harmonics, whose orders are integral multiples of the number of cranks.

§ 86. **The Inertia-Torque on the Crankshaft.**—We obtain this torque by taking the moment of the crank-pin inertia loads about the crankshaft centre. The crank-pin loads are set out in (5), Art. 81, and noting that the lever-arm for forces on the crank-pin acting along the cylinder axis is NB, while for forces in the direction at right angles to the cylinder axis it is CN, we get the crankshaft torque to be

$$-m_2k^2\ddot{\phi} \,.\, \mathrm{CN}/b - m_4\dot{\theta}^2 \,.\, \mathrm{NB} \,.\, \mathrm{CN}$$
$$+ m_4(\dot{\theta}^2 \,.\, \mathrm{NB} \,.\, \mathrm{NC} + l\ddot{\phi} \,.\, \mathrm{NC})(b - l)/l - m_4l\dot{\phi}^2 \,.\, \mathrm{NB}.$$

Substituting NB $= c \sin \phi$, and CN $= b\dot{\phi}/\dot{\theta}$, this becomes

$$m_4lb\dot{\phi}\ddot{\phi}/\dot{\theta} - (m_2k^2 + m_4l^2)\dot{\phi}\ddot{\phi}/\dot{\theta} - m_4lc(\dot{\phi}^2 \sin \phi + \dot{\theta}\dot{\phi} \sin \phi).$$

In Examples 9, 11, 12, 14, Appendix, expansions will be found for each term of this expression, and it will be seen that the crankshaft inertia-torque may be written in a series of the form

$$\dot{\theta}^2(\mathrm{B}_1 \sin \theta + \mathrm{B}_2 \sin 2\theta + \ldots + \mathrm{B}_n \sin n\theta + \ldots).$$

In a multicylinder engine all harmonics of the crankshaft inertia-torque are in balance except the harmonics, whose orders are integral multiples of the number of cranks.

§ 87. **Explosion-Torque Oscillating-Cylinder.**—The crank effort given by one cylinder throughout a complete cycle is given in table below, calculated for an engine in which the crankshaft trunnion distance is four crank lengths. From Fig. 29 we have the piston travel from the end of its stroke given by $d = b + a - c$. The crank torque is P . BN $=$ P . AB $\sin \phi$. The crank effort is P . AB $\sin \phi/\mathrm{BC} = \mathrm{P}\,\frac{c}{a} \sin \phi$. In the example, c/a is taken equal to 4, and P is obtained as in Art. 29. To calculate $\sin \phi$, observe $a/\sin \phi = b/\sin \theta$.

Crank Angle.	Cam Angle x.	b.	d.	Per Cent. Stroke.	Volume.	Absolute Pressure.	Gauge Pressure.	$\frac{c}{a} \sin \phi$.	Crank Effort.	
0	0	3	0	0	25	—	—	—	0	y_0
30	15	3·17	·17	8·5	33·5	297	282	·631	178	y_1
60	30	3·60	·60	30	55	156	141	·96	135	y_2
90	45	4·12	1·12	56	81	95	80	·97	78	y_3
120	60	4·58	1·58	79	104	68	53	·756	40	y_4
150	75	4·89	1·89	94·5	119·5	57	42	·409	17	y_5
180 to 540	90 to 270	5·00	2	100	125	54	39	0	0	y_6 to y_{18}
570	285	4·89	1·89	94·5	119·5	16	1	− ·409	− ·4	y_{19}
600	300	4 58	1·58	79	104	19	4	− ·756	— 3	y_{20}
630	315	4·12	1·12	56	81	26	11	− ·97	− 11	y_{21}
660	330	3·60	·60	30	55	44	29	− ·96	− 28	y_{22}
690	345	3·17	·17	8·5	33·5	84	69	− ·631	− 43	y_{23}

The coefficients in the series

$$E = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_{12} \cos 12x$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_{11} \sin 11x,$$

representing the crank effort as a function of the camshaft angle x, calculated from the foregoing table, are

Cosine Coefficients.		Sine Coefficients.	
$a_0 = 15{\cdot}1$			
$a_1 = 24{\cdot}4$	$a_7 = -6{\cdot}5$	$b_1 = 21{\cdot}3$	$b_7 = 9{\cdot}3$
$a_2 = 11{\cdot}4$	$b_3 = -6{\cdot}7$	$b_2 = 32{\cdot}2$	$b_8 = 6{\cdot}0$
$a_3 = -{\cdot}05$	$a_9 = -6{\cdot}1$	$b_3 = 30{\cdot}7$	$b_9 = 3{\cdot}5$
$a_4 = -5{\cdot}3$	$a_{10} = -5{\cdot}6$	$b_4 = 23{\cdot}3$	$b_{10} = 2{\cdot}5$
$a_5 = -5{\cdot}9$	$a_{11} = -5{\cdot}8$	$b_5 = 16{\cdot}7$	$b_{11} = 1{\cdot}5$
$a_6 = -5{\cdot}8$	$a_{12} = -3{\cdot}1$	$b_6 = 12{\cdot}5$	

It will be observed that the coefficients are not very different from those of the ordinary engine (Art. 64), and what has been written there will apply here with the necessary slight numerical changes.

§ 88. **Radial Oscillating-Cylinder Engine.**—For a radial oscillating-cylinder one-crank engine, it will be evident that the inertia couples on the crankshaft and the engine-frame will be identical with those for a corresponding cylinder in line type, given in Arts. 83, 84. The forces V and H at the crankshaft will for any particular cylinder be along the radius to its trunnion and at right angles to this radius respectively. Consider the *n*th order **V**-force, i.e., that along the radius. For cylinders 0, 1, 2, . . ., this force is

Cylinder 0, $F_n \cos n\theta \times 1.$
Cylinder 1, $F_n \cos n(\theta + \lambda) \times e^{i\lambda}.$
Cylinder 2, $F_n \cos n(\theta + 2\lambda) \times e^{2i\lambda}.$
. . . .
Cylinder $c - 1$, $F_n \cos n(\theta + \overline{c - 1}\lambda) \times e^{\overline{c-1}i\lambda}.$

We have multiplied the **V**-force for each cylinder (Fig. 30) by its appropriate cyclic factor, so that we can add algebraically all the terms on the right of the above table to get the resultant force due to all the cylinders. A corresponding form was investigated in Art. 50, so that it is sufficient here to restate the result there found, namely, all harmonics of **V** are in balance, except those of order n given by the formula $n = kc + 1$, where k is an integer.

The H forces are normal to their respective trunnion radii. Multiply all by i, and we obtain a set of forces along the trunnion radii. Considering the nth harmonic of the H forces, we have

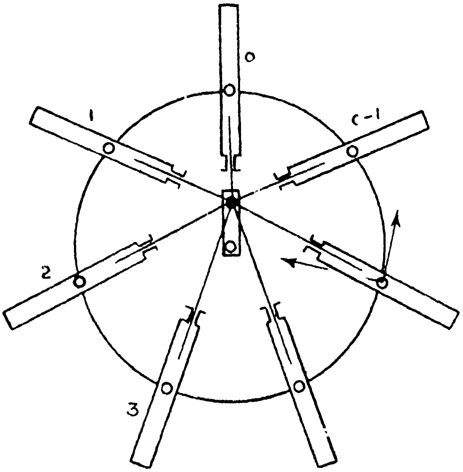

FIG. 30.—Radial oscillating cylinder engine.

iH for cylinder 0,	$A_n \sin n\theta$.
iH for cylinder 1,	$A_n \sin n(\theta + \lambda) \times e^{i\lambda}$.
iH for cylinder 2,	$A_n \sin n(\theta + 2\lambda) \times e^{2i\lambda}$.
. . . . .	. . . . .
iH for cylinder $c - 1$,	$A_n \sin n(\theta + \overline{c - 1}\lambda) \times e^{\overline{c-1}i\lambda}$.

From this table it may be shown, as in Art. 50, that all the H forces are in balance except those given by $n = kc \pm 1$.

§ 89. **Inertia-Torque and Explosion-Torque.**—The inertia-torque and explosion-couple on the crankshaft and the reverse of these torques on the engine-frame for the engine of Fig. 30 are the same as for a c-cylinder trunnion-in-line type.

CHAPTER VIII.

ROTARY TYPE OSCILLATING-CYLINDER ENGINE.

§ 90. **The Seja Engine.**—The Seja engine, exhibited at the Paris Aeronautical Show, 1920, has seven steel cylinders mounted in trunnions within a couple of heavy hoops forming the framework of the engine. The entire engine revolves around a fixed single throw crank; the connecting rods are bolted to the aluminium pistons; bore, 186 mm.; stroke, 130 mm.

§ 91. **Equivalent Dynamical System.**—Let m_1, m_2, m_3, m_4 be the masses of the equivalent dynamical system obtained as in Arts. 17, 79. The mass m_3, being at rest, does not require further consideration. The displacement of the mass m_1 at A is

$$\mathrm{BA} = z_1 = ce^{i\theta}.$$

Hence the velocity and acceleration of m_1 are respectively:

$$\dot{z}_1 = ice^{i\theta} \,.\, \dot{\theta}$$
$$\ddot{z}_1 = i^2ce^{i\theta} \,.\, \dot{\theta}^2 = - e^{i\theta} \,.\, c\omega^2.$$

The inertia resistance due to m_1 is, therefore, given by

$$m_1e^{i\theta}c\omega^2 \quad . \quad . \quad . \quad . \quad (a)$$

Denoting the displacement BQ of the mass m_2 by z_2, we have

$$\mathrm{BA} + \mathrm{AQ} = z_2 = ce^{i\theta} + ke^{i\psi}.$$
$$\therefore \dot{z}_2 = ice^{i\theta}\dot{\theta} + ike^{i\psi} \,.\, \dot{\psi}$$
$$\ddot{z}_2 = i^2cc^{i\theta}\dot{\theta}^2 + i^2ke^{i\psi}\dot{\psi}^2 + ike^{i\psi}\ddot{\psi}.$$

Multiplying $\ddot{z}_2$ by $- m_2$, we obtain the inertia resistance due to m_2. We see that it may be analysed as consisting of three parts:

A load $m_2ce^{i\theta}\dot{\theta}^2$ through m_2 parallel to BA . . . . (b)
A load $m_2ke^{i\psi} \,.\, \dot{\psi}^2$ through m_2 along AQ . . . . (c)
A load $- m_2ike^{i\psi}\ddot{\psi}$ through m_2 perpendicular to AQ . (d)

Finally, denoting the displacement BP of the mass m_4 by z_4, we have

$$\begin{aligned} z_4 &= \mathrm{BP} = \mathrm{BC} + \mathrm{CP} = a + le^{i\psi} \\ \dot{z}_4 &= ile^{i\psi} \,.\, \dot{\psi} \\ \ddot{z}_4 &= i^2 le^{i\psi}\dot{\psi}^2 + ile^{i\psi} \,.\, \ddot{\psi} \end{aligned}$$

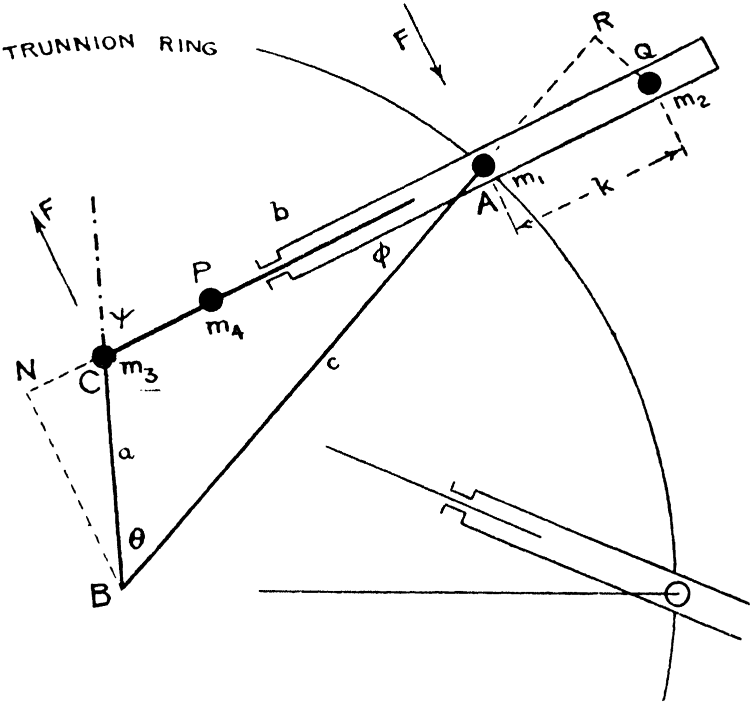

FIG. 31.—Rotary oscillating cylinder engine.

Multiplying $\ddot{z}_4$ by m_4, and reversing the signs, we get the inertia resistance due to m_4 to be :

A load $m_4 le^{i\psi}\dot{\psi}^2$ through m_4 along CP . . . . . (e)
A load $- m_4 ile^{i\psi}\ddot{\psi}$ through m_4 perpendicular to CP . (f)

§ 92. **Effects of the Inertia Loads.**—We here consider the effects of the loads (a) to (f) of last Article.

(a) This load through m_1 evidently causes a load

$$e^{i\theta} \,.\, m_1 c\dot{\theta}^2 \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

on the trunnion-bearing at A.

(*b*) The force $e^{i\theta} . m_2c\dot{\theta}^2$, through m_2 parallel to BA, can be resolved into a force along and at right angles to the cylinder. The component force along the cylinder is

$$e^{i\psi} . m_2c\dot{\theta}^2 \cos\phi \quad . \quad . \quad . \quad \text{(II.)}$$

This force causes a load of the same amount on the trunnion-bearing at A. The component of (*b*) at right angles to the cylinder is

$$- ie^{i\psi} . m_2c\dot{\theta}^2 \sin \phi.$$

It may be shown, as in Art. 79, that this component causes a load

$$- ie^{i\psi} . m_2c\dot{\theta}^2 \sin \phi . (b + k)/b \quad . \quad . \quad . \quad \text{(III.)}$$

on the trunnion-bearing at A, and a load

$$ie^{i\psi} . m_2c\dot{\theta}^2 \sin \phi . k/b \quad . \quad . \quad . \quad \text{(IV.)}$$

on the crank-pin at C.

(*d*) Similarly, it may be shown that the force $- ie^{i\psi} . m_2k\ddot{\psi}$ through m_2 perpendicular to the cylinder causes a load

$$- ie^{i\psi} . m_2k\ddot{\psi}(b + k)/b \quad . \quad . \quad . \quad \text{(V.)}$$

on the trunnion-bearing at A, and a load

$$ie^{i\psi} . m_2k\ddot{\psi}k/b \quad . \quad . \quad . \quad . \quad \text{(VI.)}$$

on the crank-pin at C.

(*f*) In like manner the force $-ie^{i\psi} . m_4l\ddot{\psi}$ through m_4, perpendicular to the rod, may be shown to cause a load

$$- ie^{i\psi} . m_4l\ddot{\psi}l/b \quad . \quad . \quad . \quad \text{(VII.)}$$

on the trunnion-bearing at A, and a load

$$- ie^{i\psi} . m_4l\ddot{\psi}(b - l)/b \quad . \quad . \quad \text{(VIII.)}$$

on the crank-pin at C.

(*c*) The force

$$e^{i\psi} . m_2k\dot{\psi}^2 \quad . \quad . \quad . \quad . \quad \text{(IX.)}$$

through m_2 along AQ causes a load on the trunnion-bearing at A of the same amount.

(*e*) The force

$$e^{i\psi} . m_4l\dot{\psi}^2 \quad . \quad . \quad . \quad . \quad \text{(X.)}$$

through m_4 along CP causes a load on the crank-pin at C of equal amount.

§ 93. **Summary of Results.**—Due to inertia, we therefore have acting on the crank-pin the loads IV., VI., VIII., and X., the resultant of which is the vector sum

$$\begin{aligned}&ie^{i\psi}[(m_2ck\dot\theta^2 \sin\phi + m_2k^2\ddot\psi)/b]\\ &- ie^{i\psi}\,.\,m_4l\ddot\psi(b-l)/b + e^{i\psi}\,.\,m_4l\dot\psi^2 \quad . \quad . \quad (1)\end{aligned}$$

Acting on the trunnion-bearing at A, we have the loads I., II., IX., III., V., and VII., the vector sum of which is

$$\begin{aligned}&e^{i\theta}\,.\,m_1c\dot\theta^2 + e^{i\psi}(m_2c\dot\theta^2\cos\phi + m_2k\dot\psi^2)\\ &- ie^{i\psi}[m_2(c\dot\theta^2\sin\phi + k\dot\psi)(b+k)/b + m^4l^2\ddot\psi/b] \quad . \quad (2)\end{aligned}$$

§ 94. **Effect of Inertia-Loads on Trunnion-Bearing.**—The effect of the trunnion-bearing inertia-loads is to give a set of equal loads on the propeller-shaft-bearing at B, together with a torque on the propeller-shaft. To obtain the propeller-shaft inertia-torque, we take the sum of the moments about B of each inertia-load at A.

§ 95. **Replacement of Crank-pin Inertia-Loads by Equivalent Force-System.**—The effect of the crank-pin inertia-loads in causing vibration of the engine-frame is the same as the combined effects of an equal set of loads acting through the propeller-shaft-bearing B, and a couple on the engine-frame, the moment of the couple being the sum of the moments about B of the inertia loads on the crank-pin. The complete dynamic effect of the engine motion is thus represented by

(1) A force on the engine-frame acting through the centre of the propeller-shaft.
(2) A couple on the engine-frame.
(3) A couple on the propeller-shaft.

§ 96. **The Inertia-Force on the Engine-Frame through Propeller-Shaft Centre.**—From Arts, 94, 95 we see that the inertia-force on the engine-frame, acting through the propeller-shaft centre B, is the sum of the loads acting on both crank-pin and trunnion-bearing, that is, the loads I. to X. set out in Art. 93. Summing these loads, we get the inertia-force on the engine-frame to be

$$\begin{aligned}\mathrm{F} &= e^{i\theta}\,.\,m_1c\dot\theta^2 + e^{i\psi}\,.\,m_2c\dot\theta^2(\cos\phi - i\sin\phi)\\ &\quad + e^{i\psi}(m_2k + m_4l)(\dot\psi^2 - i\ddot\psi).\end{aligned}$$

Since the second term here can be written

$$e^{i\psi}\,.\,e^{-i\phi}\,.\,m_2c\dot\theta^2 = e^{i(\psi-\phi)}\,.\,m_2c\dot\theta^2 = e^{i\theta}\,.\,m_2c\dot\theta^2,$$

the sum of the first and second terms in the expression for F is

$$e^{i\theta}\,(m_1 + m_2)c\dot{\theta}^2,$$

which is evidently equivalent to the dynamic effect of a mass $m_1 + m_2$ at, and rotating with, the trunnion-bearing. In a multicylinder engine having its cylinders evenly spaced on the trunnion-ring, this portion of the engine-frame inertia-force is in collective balance.

Since $\psi = \theta + \phi$, we have $e^{i\psi} = e^{i(\theta+\phi)} = e^{i\theta} . e^{i\phi}$, and also we have $\dot{\psi} = \dot{\theta} + \dot{\phi}$; $\ddot{\psi} = \ddot{\theta}$; so that

$$\begin{aligned} e^{i\psi}(\dot{\psi}^2 - i\ddot{\psi}) &= e^{i\theta} . e^{i\phi}(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi} - i\ddot{\phi}) \\ &= e^{i\theta}(\cos\phi + i\sin\phi)(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi} - i\ddot{\phi}) \\ = e^{i\theta}\{\dot{\theta}^2\cos\phi + (\dot{\phi}^2\cos\phi &+ \ddot{\phi}\sin\phi) - i(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi) \\ &+ 2\dot{\theta}\dot{\phi}\cos\phi + i\dot{\theta}^2\sin\phi + 2i\dot{\theta}\dot{\phi}\sin\phi\} \end{aligned}$$

Referring now to the Appendix, Examples 10, 11, we find that this expression can be written in the form

$$\begin{aligned} e^{i\theta} . \dot{\theta}^2[&A_1\cos\theta + A_2\cos 2\theta + \ldots + A_n\cos n\theta + \ldots \\ &+ i(B_1\sin\theta + B_2\sin 2\theta + \ldots + B_n\sin n\theta + \ldots)] \end{aligned}$$

Multiplying this by $(m_2k + m_4l)$, we get the remaining portion of the engine-frame inertia-force.

The sum of the terms involving A_n and B_n can, as shown in Art. 69, be written in the form

$$Ke^{(n+1)i\theta} + Le^{-(n-1)i\theta}.$$

For a c-cylinder engine we therefore have unbalanced those inertia-force harmonics whose coefficients involve the subscripts $kc \pm 1$, where k is an integer. This result follows from the investigation given in Art. 69 The Seja seven-cylinder engine has unbalanced 7th, 14th, . . . inertia-force harmonics.

§ 97. **Inertia-Torque on Engine Frame.**—As stated in Art. 95, the moment of the engine-frame inertia-couple is the sum of the moments about B of the inertia-loads on the crank-pin C. The crank-pin inertia-loads are set out in (1), Art. 93, and taking the moment of (1) about the propeller-shaft bearing B, we get the couple on the engine-frame to be

$$\begin{gathered} \mathrm{NC}[m_2ck\dot{\theta}^2\sin\phi + m_2k^2\ddot{\psi} - m_4l\ddot{\psi}(b - l)]/b \\ + \mathrm{NB} . m_4l\dot{\psi}^2. \end{gathered}$$

The velocity of A normal to CA is $b\dot{\psi}$. It is also $c\dot{\theta}\cos\phi$, the resolute perpendicular to CA of the velocity of A. Hence

$$b\dot{\psi} = \mathrm{C}\dot{\theta}\cos\phi = \mathrm{AN}\,.\,\dot{\theta}$$
$$\therefore\ \mathrm{AN} = b\dot{\psi}/\dot{\theta} = b(\dot{\theta} + \dot{\phi})/\dot{\theta} = b(1 + \dot{\phi}/\dot{\theta}).$$

Since NC = AN − b, we have

$$\mathrm{NC} = a\cos\psi = b\dot{\phi}/\dot{\theta}.$$

Making use of these relations, the above expression for the engine-frame torque reduces to

$$m_4 l\dot{\psi}^2 c\sin\phi - m_4 l\ddot{\psi}a\cos\psi + m_2 c\dot{\theta}k\phi\sin\phi + \dot{\theta}^{-1}(m_2k^2 + m_4l^2)\dot{\phi}\ddot{\phi}.$$

Since $c\sin\phi = a\sin\psi$, the first two terms can be written

$$m_4 al(\dot{\psi}^2\sin\psi - \ddot{\psi}\cos\psi).$$

Again, since $\sin\phi = \rho\sin\psi$, successive differentiation gives

$$\dot{\phi}\cos\phi = \rho\dot{\psi}\cos\psi$$
$$\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi = \rho(\ddot{\psi}\cos\psi - \psi^2\sin\psi).$$

Further, $a = \rho c$.

Hence the total engine-frame torque is

$$-\,m_4 lc(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi) + m_2 c\dot{\theta}k\dot{\phi}\sin\phi + \dot{\theta}^{-1}(m_2k^2 + m_4l^2)\phi\ddot{\phi}.$$

Reference to Examples 9, 10, 11, Appendix, will show that this torque can be written in the form

$$\dot{\theta}^2(\mathrm{C}_1\sin\theta + \mathrm{C}_2\sin 2\theta + \ .\ .\ .\ + \mathrm{C}_n\sin n\theta + \ .\ .\ .).$$

In a multicylinder engine the n'th harmonic of this torque is unbalanced only when n is an integral multiple of the number of cylinders. The magnitude of the unbalanced nth harmonic of the engine-frame torque is $\dot{\theta}^2\mathrm{C}_n\sin n\theta$ per cylinder.

§ 98. **Inertia-Torque on Propeller Shaft.**—From Art. 94 we have the propeller-shaft inertia-torque given by taking the moment of (2), Art. 93, about the propeller-shaft axis. This moment is

$$\mathrm{BN}(m_2c\dot{\theta}^2\cos\phi + m_2k\dot{\psi}^2)$$
$$-\ \mathrm{AN}[m_2(c\dot{\theta}^2\sin\phi + k\ddot{\psi})(b + k)/b + m_4l^2\ddot{\psi}/b].$$

Making use of the relations established in the preceding article, this expression reduces to

$$m_2kc\{\dot{\phi}^2\sin\phi - \ddot{\phi}\cos\phi\} + m_2kc\dot{\theta}\dot{\phi}\sin\phi - (m_2k^2 + m_4l^2)\ddot{\phi}$$
$$-\ \dot{\theta}^{-1}(m_2k^2 + m_4l^2)\dot{\phi}\ddot{\phi}.$$

Reference to Examples 8, 9, 10, 11, Appendix, will show that this torque can be written in the form

$$\dot{\theta}^2(\mathrm{D}_1\sin\theta + \mathrm{D}_2\sin 2\theta + \ .\ .\ .\ + \mathrm{D}_n\sin n\theta + \ .\ .\ .).$$

The coefficients D_1, D_2, . . . can be readily calculated from the coefficients given in the Appendix Examples above quoted.

In a c-cylinder engine the nth harmonic of the propeller-shaft inertia-torque is unbalanced only when $n = kc$, where k is an integer. The total propeller-shaft inertia-torque is $c \,.\, \Sigma D_{kc} \sin kc\theta$. Thus the seven-cylinder Seja engine has an inertia-torque on the propeller-shaft given by $7(D_7 \sin 7\theta + D_{14} \sin 14\theta + \ldots)$.

§ 99. **Explosion-Torque Rotary-Oscillator.**—The explosion torque of the rotary type oscillating-cylinder is exactly the same as for the fixed trunnion-ring type given in Art. 89. The Seja has the seventh harmonic explosion torque unbalanced. The period of this fluctuating torque, it must be remembered, is one-seventh the "camshaft" period, and is therefore two-sevenths the engine period.

This gives $3\theta = -13\frac{1}{2}^\circ + n180^\circ$, from which $\theta = -4\frac{1}{2}^\circ$; $55\frac{1}{2}^\circ$; $115\frac{1}{2}^\circ$; $175\frac{1}{2}^\circ$; $235\frac{1}{2}^\circ$; $295\frac{1}{2}^\circ$. As the angle θ changes from $-4\frac{1}{2}^\circ$ to $55\frac{1}{2}^\circ$, the couple C is negative and energy is being absorbed from the flywheel. As the crank turns from $55\frac{1}{2}^\circ$ to $115\frac{1}{2}^\circ$, the couple C is positive, and it therefore adds energy to the flywheel during this period. From $115\frac{1}{2}^\circ$ to $175\frac{1}{2}^\circ$ C is again negative, and the flywheel speed falls until the couple becomes positive again for crank angles between $175\frac{1}{2}^\circ$ and $235\frac{1}{2}^\circ$. From $235\frac{1}{2}^\circ$ to $295\frac{1}{2}^\circ$ C is negative; from $295\frac{1}{2}^\circ$ to $-4\frac{1}{2}^\circ$ C is positive. The maximum speed of the flywheel will be immediately after a period of positive accelerating couple, that is, at the angles $115\frac{1}{2}^\circ$, $235\frac{1}{2}^\circ$, $-4\frac{1}{2}^\circ$. The minimum speed of the flywheel corresponds to the angles $55\frac{1}{2}^\circ$, $175\frac{1}{2}^\circ$, $295\frac{1}{2}^\circ$.

The fluctuation in speed $\delta\Omega_1$, from minimum to maximum, is given by

$$\delta\Omega_1 = \frac{1}{I\omega_o}\int_{\theta = 55\frac{1}{2}^\circ}^{\theta = 115\frac{1}{2}^\circ} C \, . \, d\theta$$

$$= \frac{g}{I\omega_o}\left|\frac{281}{3}\cos 3\theta - \frac{68}{3}\sin 3\theta\right|_{55\frac{1}{2}^\circ}^{115\frac{1}{2}^\circ}$$

$$= \frac{6163}{I\omega_o} = \frac{6163}{33 \times 125} = 1{\cdot}5 \text{ radian per sec.}$$

At any instant between $\theta = 55\frac{1}{2}^\circ$ and $\theta = 115\frac{1}{2}^\circ$ $\delta\Omega$ is given by

$$\delta\Omega = \frac{32}{I\omega_o}\int(-281\sin 3\theta - 68\cos 3\theta)d\theta$$

$$= \frac{32}{I\omega_o}\left(\frac{281}{3}\cos 3\theta - \frac{68}{3}\sin 3\theta\right) + K$$

where K is an arbitrary constant. Measuring $\delta\Omega$ from the mean speed ω_o, we have the average value of $\delta\Omega$ zero, so that K must be zero. Hence we have

$$\delta\Omega = \omega_r = \frac{32}{3I\omega_o}(281\cos 3\theta - 68\sin 3\theta).$$

ω_r is zero when $281\cos 3\theta = 68\sin 3\theta$, or $\tan 3\theta = 4{\cdot}13$, from which $3\theta = 76\frac{1}{2}^\circ + n180^\circ$, or $\theta = 25\frac{1}{2}^\circ$, $85\frac{1}{2}^\circ$, $145\frac{1}{2}^\circ$, $205\frac{1}{2}^\circ$, $265\frac{1}{2}^\circ$, $325\frac{1}{2}^\circ$.

As the crank turns from $85\frac{1}{2}^\circ$ to $145\frac{1}{2}^\circ$ the actual flywheel has a greater speed than the fictitious wheel; from $145\frac{1}{2}^\circ$ to $205\frac{1}{2}^\circ$ the actual wheel is slower than the fictitious wheel. The actual wheel will have its maximum advance on the fictitious wheel at the end of a period for which ω_r is positive, that is, at the end of the crank period $85\frac{1}{2}^\circ$ to $145\frac{1}{2}^\circ$. This angular displacement μ_1 is given by

$$\begin{aligned}\mu_1 &= \int_{\theta = 85\frac{1}{2}^\circ}^{\theta = 145\frac{1}{2}^\circ} \omega_r dt \\ &= \int \frac{32}{3I\omega_0}(281 \cos 3\theta - 68 \sin 3\theta) d\theta \,.\, \frac{dt}{d\theta} \\ &= \frac{32}{9I\omega_0{}^2} \Big| 281 \sin 3\theta + 68 \cos 3\theta \Big|_{\theta = 85\frac{1}{2}^\circ}^{\theta = 145\frac{1}{2}^\circ} \\ &= \cdot 004 \text{ radian.}\end{aligned}$$

Example 1.—Work preceding example for a four-cylinder petrol engine, neglecting all torque harmonics other than secondaries.

Example 2.—Find the moment of inertia of flywheel for the six-cylinder petrol engine of § 102, if the fluctuation in speed from minimum to maximum is not to exceed 1·5 per cent. of the mean speed which is 1000 revolutions per minute.

Example 3.—Find the moment of inertia of flywheel for the four-cylinder petrol engine of Ex. 1, the permissible fluctuation in speed being 3 per cent. of the mean speed which is 1200 revolutions per minute.

CHAPTER X.

DESIGN OF COPLANAR CRANK SYSTEMS WITH SPECIFIED BALANCE.

§ 103. **The Equations.**—In the foregoing chapters we have investigated the balance of engines of given design. We now propose to investigate the converse problem, namely, the synthesis of crank systems to satisfy given conditions of balance. In the crank arrangement (Fig. 32), θ_1, θ_2, θ_3, . . . are the angles between the crank No. 0 and the cranks 1, 2, 3 . . . The reciprocating masses associated with the cranks 0, 1, 2, 3, . . . are M_0, M_1, M_2, M_3, . . . and the corresponding crank radii are r_0, r_1, r_2, r_3. . . . Let $M_0r_0 = m_0$; $M_1r_1 = m_1$; $M_2r_2 = m_2$; etc. Let the cranks rotate with uniform angular velocity Ω. Consider the system at the instant on which

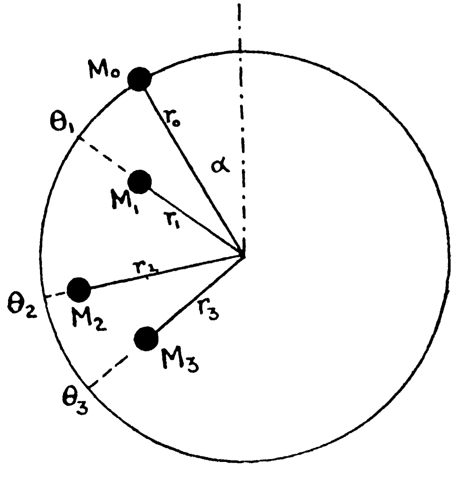

FIG. 32.—Cranks operating reciprocating masses M_0, M_1, M_2, . . .

crank No. 0 makes an angle α with the vertical. The nth order inertia force for a cylinder in line engine is from Art. 18 :—

$$\begin{aligned} F_n &= \Omega^2 p_n\{m_0 \cos n\alpha + m_1 \cos n(\alpha+\theta_1) + m_2 \cos n(\alpha+\theta_2) + \ldots\} \\ &= \cdot 5\Omega^2 p_n[m_0\{e^{ni\alpha} + e^{-ni\alpha}\} + m_1\{e^{ni(\alpha+\theta_1)} + e^{-ni(\alpha+\theta_1)}\} + \ldots] \\ &= \cdot 5\Omega^2 p_n e^{ni\alpha}[m_0 + m_1 e^{ni\theta_1} + m_2 e^{ni\theta_2} + \ldots \\ &\qquad + e^{-2ni\alpha}(m_0 + m_1 e^{-ni\theta_1} + m_2 e^{-ni\theta_2} + \ldots)] \end{aligned}$$

To have F_n zero for all values of α, we therefore must have

$$m_0 + m_1 e^{ni\theta_1} + m_2 e^{ni\theta_2} + \ldots = 0 \quad . \quad (1A)$$

and

$$m_0 + m_1 e^{-ni\theta_2} + m_1 e^{-ni\theta_2} + \ldots = 0 \quad . \quad (2A)$$

If these two equations be divided throughout by m_0, and if w_1 be written for m_1/m_0, w_2 for m_2/m_0, etc., we get

$$1 + w_1 e^{ni\theta_1} + w_2 e^{ni\theta_2} + \ldots = 0 \quad . \quad . \quad (1)$$

$$1 + w_1 e^{-ni\theta_1} + w_2 e^{-ni\theta_2} + \ldots = 0 \quad . \quad . \quad (2)$$

When these two equations are satisfied, we have the nth harmonics of the inertia forces in balance. Putting $n = 1$, we get the conditions for balance of the primary inertia forces to be

$$1 + w_1 e^{i\theta_1} + w_2 e^{i\theta_2} + \ldots = 0 \quad . \quad . \quad (3)$$

$$1 + w_1 e^{-i\theta_1} + w_2 e^{-i\theta_2} + \ldots = 0 \quad . \quad . \quad (4)$$

Again, putting $n = 2$ in (1) and (2), we get the conditions for balance of the secondary inertia forces to be

$$1 + w_1 e^{2i\theta_1} + w_2 e^{2i\theta_2} + \ldots = 0 \quad . \quad . \quad (5)$$

$$1 + w_1 e^{-2i\theta_1} + w_2 e^{-2i\theta_2} + \ldots = 0 \quad . \quad . \quad (6)$$

For balance of both primary and secondary forces, we require the equations (3), (4), (5), (6) to be satisfied.

In future we shall find it more convenient to write a for $e^{i\theta_1}$, b for $e^{i\theta_2}$, c for $e^{i\theta_3}$, d for $e^{i\theta_4}$, etc. Making these changes, the equations for primary and secondary balance become

$$\begin{aligned} 1 + aw_1 + bw_2 + cw_3 + \ldots &= 0. \\ 1 + \frac{1}{a} w_1 + \frac{1}{b} w_2 + \frac{1}{c} w_3 + \ldots &= 0. \\ 1 + a^2 w_1 + b^2 w_2 + c^2 w_3 + \ldots &= 0 \\ 1 + \frac{1}{a^2} w_1 + \frac{1}{b^2} w_2 + \frac{1}{c^2} w_3 + \ldots &= 0. \end{aligned}$$

§ 104. **The Variables.**—Examination of the above equations shows that for an engine having c cranks we have $c - 1$ angles (θ's) and the same number of w's, or, altogether, $2c - 2$ variables. For primary balance, since two of these variables must be left unassigned to be determined by the solution of the equations (3), (4), the number of variables to which arbitrary values may be assigned in the design is $(2c - 2) - 2$, or $2c - 4$. Thus, when $c = 2$, we are not at liberty to fix θ_1 or w_1. These quantities are given by the solution ot (3) and (4). Again, for complete primary and secondary balance, we require the four equations (3), (4), (5), (6) satisfied, so that four of the variables must be left unassigned. The least number of cranks which permit of four variables being left unassigned is given by $2c - 2 = 4$, or $c = 3$. We cannot then obtain primary and secondary balance with fewer than three cranks. In the design of a four-crank for primary balance we can fix four of the variables, the remaining two being then determined from the solution of the equations (3) and (4) (Art. 103). In the design of a four-crank for both primary and secondary balance, we are at liberty to select at random but two of the six variables occurring; the remaining four variables being obtained from the solution of the equations (3), (4), (5), (6) (Art. 103). It may be noted that, given any coplanar crank system, we can secure its primary and secondary balance by the addition of two extra cranks. The angles and crank masses of the two additional cranks give four variables, which are determinable from the four equations for primary and secondary balance.

§ 105. **Two-Crank Engine.**—The variables here are θ_1, the angle between the cranks, and w_1, the mass attached to crank 1, the mass attached to crank 0 being taken as unity. For balance of the primary forces we require

$$1 + aw_1 = 0$$
$$1 + \frac{1}{a}w_1 = 0.$$

Eliminating w_1, we get, in the notation of determinants,

$$\begin{vmatrix} 1 & a \\ 1 & \frac{1}{a} \end{vmatrix} = 0$$

The determinant on multiplication by a becomes

$$\begin{vmatrix} 1 & a^2 \\ 1 & 1 \end{vmatrix}$$

the expansion of which is $1 - a^2$. We therefore have

$$1 - a^2 = 0; \qquad \text{giving } a = \pm 1$$

Taking $a = 1$, and substituting in the original equations, we find $w = -1$, which clearly is inadmissible as a solution. Taking $a = -1$, we get $w_1 = 1$, which is possible. Since $e^{i\theta_1} = a = -1$, we must have $\theta_1 = \pi$. Further, since $w_1 = 1$, the crank masses are equal. The crank system is therefore as shown in Fig. 33. From Art. 18 it follows that this system is not in secondary balance.

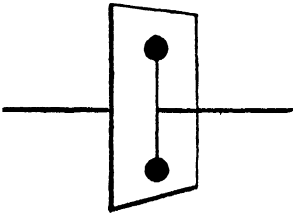

FIG. 33.—Reciprocating masses in primary balance.

§ 106. **Three-Crank Engine.**—Denoting the cranks by 0, 1, 2, let cranks 1 and 2 make angles θ_1 and θ_2 with crank 0. The equations for primary and secondary balance of a three-crank system are from Art. 103.

$$1 + aw_1 + bw_2 = 0 \quad . \quad . \quad . \quad (1)$$

$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 = 0 \quad . \quad . \quad . \quad (2)$$

$$1 + a^2w_1 + b^2w_2 = 0 \quad . \quad . \quad . \quad (3)$$

$$1 + \frac{1}{a^2}w_1 + \frac{1}{b^2}w_2 = 0 \quad . \quad . \quad . \quad (4)$$

It will be noted that as the number of variables in the above four equations is 4, one solution of the set is possible. Eliminating the w's from (1), (2), (3), we get the determinantal equation :

$$\Delta = \begin{vmatrix} 1 & a & b \\ 1 & \frac{1}{a} & \frac{1}{b} \\ 1 & a^2 & b^2 \end{vmatrix} = 0$$

The determinant Δ on multiplication by ab gives

$$\begin{vmatrix} 1 & a^2 & b^2 \\ 1 & 1 & 1 \\ 1 & a^3 & b^3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 1 \\ 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \end{vmatrix}$$

The expansion of this determinant, when written in factor form is

$$-(a - b)(b - 1)(a - 1)(a + b + ab).$$

A solution of $\Delta = 0$, consistent with cranks not being coincident, is therefore

$$a + b + ab = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Again, by eliminating the w's from (1), (2), (4), we get

$$\begin{vmatrix} 1 & a & b \\ 1 & \frac{1}{a} & \frac{1}{b} \\ 1 & \frac{1}{a^2} & \frac{1}{b^2} \end{vmatrix} = 0$$

which gives

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^3 & b^3 \end{vmatrix} = 0$$

The expansion of the determinant gives

$$(a - b)(b - 1)(c - 1)(a + b + 1) = 0.$$

Here again a solution consistent with cranks not being coincident is

$$1 + a + b = 0 \quad . \quad . \quad . \quad . \quad (6)$$

Hence we have

$$a + b + ab = 0 \quad . \quad . \quad . \quad (5)$$
$$1 + a + b = 0 \quad . \quad . \quad . \quad (6)$$

Solving this pair of simultaneous equations, we get $ab = 1$; $a + b = -1$.

$$\therefore (a + b)^2 - 4ab = 1 - 4 = -3$$
$$\therefore a - b = \pm\sqrt{3}$$
$$a + b = -1$$
$$\therefore a = \tfrac{1}{2}(-1 \pm \sqrt{-3})\,;\quad b = \tfrac{1}{2}(-1 \mp \sqrt{-3}),$$

that is, $e^{i\theta_1} = \frac{1}{2}(-1 + \sqrt{-3})$ and $e^{i\theta_2} = \frac{1}{2}(-1 - \sqrt{-3})$. Hence $\theta_1 = 120°$ and $\theta_2 = 240°$.

Multiplying equation (2) by ab, we get

$$ab + bw_1 + aw_2 = 0$$
$$\therefore 1 + bw_1 + aw_2 = 0 \text{ (since } ab = 1).$$

From (1) we have $1 + aw_1 + bw_2 = 0$, so that on subtraction we get

$$w_1(a - b) + w_2(b - a) = 0.$$

Also, by addition,

$$2 + (a + b)w_1 + (a + b)w_2 = 0.$$

From the former equation we have $w_1 = w_2$. Substituting this result in the latter equation, and remembering that $a + b = -1$, we get

$$2 - w_1 - w_1 = 0.$$
$$\therefore w_1 = 1.$$

Hence $w_2 = w_1 = 1$, so that the crank masses are all equal. The crank system is, therefore, as shown in Fig. 34.

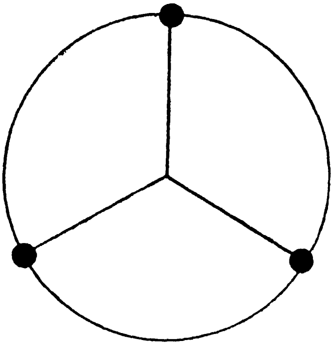

FIG. 34.—Reciprocating masses with primary and secondary balance.

§ 107. **Four-Crank System.**—The equations for primary and secondary balance of a four-crank system are from Art. 103:

$$1 + aw_1 + bw_2 + cw_3 = 0$$
$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 = 0$$
$$1 + a^2w_1 + b^2w_2 + c^2w_3 = 0$$
$$1 + \frac{1}{a^2}w_1 + \frac{1}{b^2}w_2 + \frac{1}{c^2}w_3 = 0.$$

As there are altogether six variables in this set of equations, we are at liberty to assign values to two of them, leaving the remaining four to be determined from the solution. Eliminating the w's we get

$$\begin{vmatrix} a & b & c & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & 1 \\ a^2 & b^2 & c^2 & 1 \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} & 1 \end{vmatrix} = 0$$

whence

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & a^3 & a^4 \\ 1 & b & b^3 & b^4 \\ 1 & c & c^3 & c^4 \end{vmatrix} = 0$$

Development of the determinant gives

$$\begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - 1 \begin{vmatrix} 1 & a^3 & a^4 \\ 1 & b^3 & b^4 \\ 1 & c^3 & c^4 \end{vmatrix} + 1 \begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} - 1 \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

Expanding now, we get

$$\begin{aligned}&(a - b)(b - c)(c - a)(ab + bc + cc)abc\\ &- (a - b)(b - c)(c - a)(a^2b^2 + b^2c^2 + c^2a^2 + a^2bc + ab^2c + abc^2)\\ &+ (a - b)(b - c)(c - a)(a^2 + b^2 + c^2 + ab + bc + ca)\\ &- (a - b)(b - c)(c - a)(a + b + c).\end{aligned}$$

Hence

$$\begin{aligned}a^2 + b^2 + c^2 + ab + bc + ca - a - b - c - a^2b^2 - b^2c^2 - c^2a^2\\ - a^2bc - ab^2c - abc^2 + a^2b^2c + ab^2c^2 + a^2bc^2 = 0.\end{aligned}$$

From the original determinant it will be seen that $(a - 1)(b - 1)(c - 1)$ is a factor of the expression on the left. The remaining factor is readily found to be

$$a + b + c + ab + bc + ca.$$

Hence a solution of the determinantal equation consistent with the cranks being all different is

$$a + b + c + ab + bc + ca = 0.$$

For primary and secondary balance the crank angles must satisfy this relation. Given two of the angles, the remaining angle can be found from

$$a + b + c + ab + bc + ca = 0,$$

or, in other words, from

$$e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_1} . e^{i\theta_2} + e^{i\theta_2}e^{i\theta_3} + e^{i\theta_3}e^{i\theta_1} = 0.$$

For example, given a and b, to find c we have $-c = (a + b + ab)/(1 + a + b)$. We therefore find the vector representing the numerator and the vector representing the denominator. The angle between these two vectors is $\theta_3 - \pi$. To obtain the vector $a + b + ab$, we have $a = e^{i\theta_1}$; $b = e^{i\theta_2}$; $ab = e^{i\theta_1} . e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$. We therefore draw OA (Fig. 35), making an angle θ_1 with the vertical (standard direction), from its end A draw AB, making angle θ_2 with vertical, and from B draw BC, making angle $\theta_1 + \theta_2$ with vertical. OA, AB, BC are each made one unit long. The vector OC gives the sum $a + b + ab$. Let OC make an angle λ with the vertical; then $\text{OC} = me^{i\lambda}$. Next, to obtain the vector $a + b + 1$, we draw OA and AB as before, and from B we draw BD vertically one unit long. The vector OD gives the sum $a + b + 1$. Let OD make an angle

μ with the vertical; then OD $= ne^{i\mu}$. Since OC/OD $= -c$ $= -e^{i\theta_3} = e^{i\theta_3}e^{-i\pi} = e^{i(\theta_3 - \pi)}$, we have

$$me^{i\lambda}/ne^{i\mu} = \frac{m}{n}e^{i(\lambda-\mu)} = e^{i(\theta_3-\pi)}.$$

Hence $\lambda - \mu + \pi = \theta_3$ (also $m = n$).

If CO be produced to C′, it is evident that the angle DOC′ $= \theta_3$. Further, if OB be joined, θ_3 is given by twice angle DOB plus 180°.

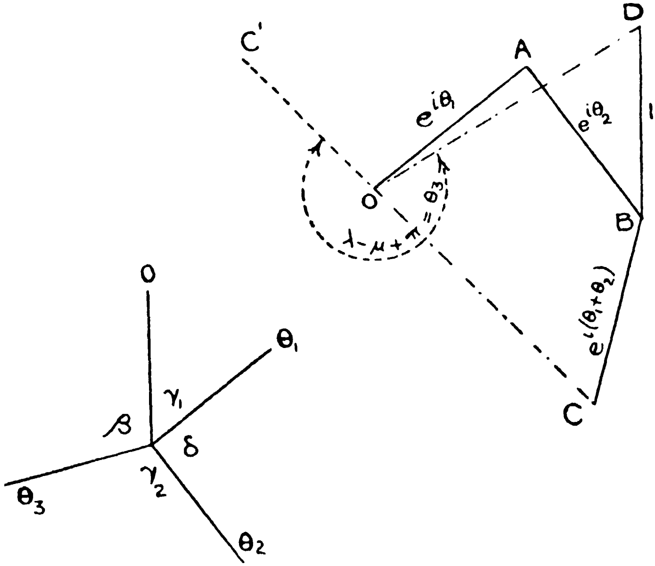

FIG. 35.—Graphical solution of $a + b + c + ab + bc + ca = 0$ given a and b.

§ 108. **Trigonometrical Relation between Angles.**—If the equation

$$a + b + c + ab + bc + ca = 0$$

be divided throughout by $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$, we get

$$a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$$
$$+ a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}} = 0.$$

$$\therefore e^{\frac{i\theta_1}{2}} . e^{\frac{-i\theta_2}{2}} . e^{\frac{-i\theta_3}{2}} + e^{\frac{-i\theta_1}{2}} . e^{\frac{i\theta_2}{2}} . e^{\frac{-i\theta_3}{2}} + e^{\frac{-i\theta_1}{2}} . e^{\frac{-i\theta_2}{2}} . e^{\frac{i\theta_3}{2}}$$
$$+ e^{\frac{i\theta_1}{2}} . e^{\frac{i\theta_2}{2}} . e^{\frac{-i\theta_3}{2}} + e^{\frac{-i\theta_1}{2}} . e^{\frac{i\theta_2}{2}} . e^{\frac{i\theta_3}{2}} + e^{\frac{i\theta_1}{2}} . e^{\frac{-i\theta_2}{2}} . e^{\frac{i\theta_3}{2}} = 0.$$

$$\therefore\ e^{\frac{1}{2}i(\theta_1-\theta_2-\theta_3)}+e^{-\frac{1}{2}i(\theta_1-\theta_2-\theta_3)}+e^{\frac{1}{2}i(\theta_1-\theta_2+\theta_3)}$$
$$+e^{-\frac{1}{2}i(\theta_1-\theta_2+\theta_3)}+e^{\frac{1}{2}i(\theta_1+\theta_2-\theta_3)}+e^{-\frac{1}{2}i(\theta_1+\theta_2-\theta_3)}=0$$

$$\therefore \cos\tfrac{1}{2}(\theta_1-\theta_2-\theta_3)+\cos\tfrac{1}{2}(\theta_1-\theta_2+\theta_3)$$
$$+\cos\tfrac{1}{2}(\theta_1+\theta_2-\theta_3)=0$$

This on expansion gives the tangent equation

$$3+\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2+\tan\tfrac{1}{2}\theta_2\tan\tfrac{1}{2}\theta_3+\tan\tfrac{1}{2}\theta_3\tan\tfrac{1}{2}\theta_1=0.$$

In Fig. 35, if $\gamma_1=\theta_1$; $\gamma_2=\theta_3-\theta_2$; $\delta=\theta_2-\theta_1$; $\beta=2\pi-\theta_3$ we have

$$\begin{aligned}\gamma_1+\gamma_2&=\theta_1-\theta_2+\theta_3\\ \gamma_2-\gamma_1&=-\theta_1-\theta_2+\theta_3\\ \beta-\delta&=2\pi-\theta_3-\theta_2+\theta_1.\end{aligned}$$

Substituting these values in the cosine equation above we get

$$\cos\tfrac{1}{2}(\gamma_2+\gamma_1)+\cos\tfrac{1}{2}(\gamma_2-\gamma_1)=\cos\tfrac{1}{2}(\beta-\delta)$$

$$\therefore\ 2\cos\frac{\gamma_2}{2}\cos\frac{\gamma_1}{2}=\cos\frac{\beta-\delta}{2}$$

If $\beta=\delta$ this becomes

$$\cos\frac{\gamma_2}{2}\cos\frac{\gamma_1}{2}=\frac{1}{2}.$$

If a square be inscribed in the crank circle, and a circle be passed through the middle points of its opposite sides, the tangents to this circle, which are parallel to the other sides of the square, cut the crank circle in points which give crank-pins satisfying the angular relation for a four-crank in primary and secondary balance. From Fig. 36, we have

$$EO\ .\ OF=OH^2=\tfrac{1}{2}OJ^2=\tfrac{1}{2}OB^2.$$
$$\therefore\ \frac{EO}{OB}\cdot\frac{OF}{OD}=\tfrac{1}{2},$$

that is,

$$\cos\tfrac{1}{2}\,AOB\ .\ \cos\tfrac{1}{2}\,COD=\tfrac{1}{2}.$$

It is interesting to note that, as shown in the lower diagrams (Fig. 36), the three-crank at 120°, and four-crank at 90°, are limiting cases given by this construction.

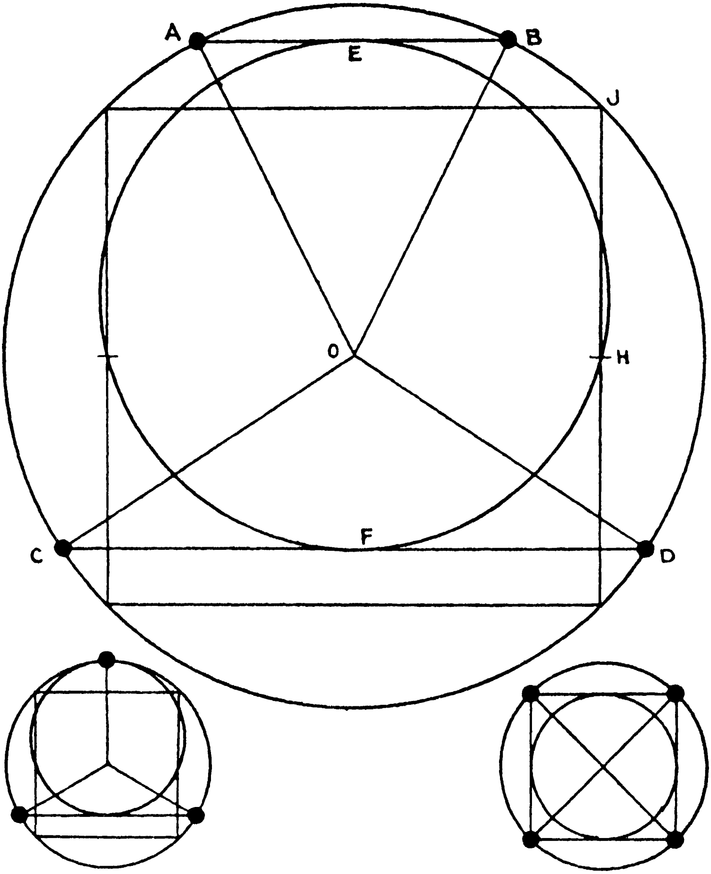

FIG. 36.—Construction giving cranks in balance for primary and secondary forces.

§ 109. **The Reciprocating Masses.**—The four equations of Art. 107 become

$$aw_1 + bw_2 + cw_3 = -1$$

$$\frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 = -1$$

$$a^2 w_1 + b^2 w_2 + c^2 w_3 = -1$$

$$\frac{1}{a^2} w_1 + \frac{1}{b^2} w_2 + \frac{1}{c^2} w_3 = -1,$$

from which

$$w_1 = \begin{vmatrix} -1 & b & c \\ -1 & \dfrac{1}{b} & \dfrac{1}{c} \\ -1 & b^2 & c^2 \end{vmatrix} \div \begin{vmatrix} a & b & c \\ \dfrac{1}{a} & \dfrac{1}{b} & \dfrac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{bc}\begin{vmatrix} 1 & 1 & 1 \\ 1 & b^2 & c^2 \\ 1 & b^3 & c^3 \end{vmatrix} \div \frac{-1}{abc}\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \frac{1}{bc}(1-b)(1-c)(b-c)(b+c+bc) \div \frac{1}{abc}(a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\therefore w_1 = \frac{a(1-b)(1-c)(b+c+bc)}{(a-b)(c-a)(ab+bc+ca)}.$$

Remembering that $a + b + c + ab + bc + ca = 0$, we have $b + c + bc = -(a + ab + ac) = -a(1 + b + c)$, and also we have $ab + bc + ca = -(a + b + c)$.

Hence $$w_1 = \frac{a^2(1-b)(1-c)(1+b+c)}{(a-b)(c-a)(a+b+c)}.$$

In like manner we obtain

$$w_2 = \frac{b^2(1-c)(1-a)(1+c+a)}{(b-c)(a-b)(a+b+c)}$$

$$w_3 = \frac{c^2(1-a)((1-b)(1+a+b)}{(c-a)(b-c)(a+b+c)}.$$

The vector expressions on the right may be evaluated graphically. Since, however, w_1 is a non-directed quantity, it is given by the modulus of the vector expression. We therefore obtain w_1, (w_2, w_3) by writing for each factor its modulus and evaluating the resulting expression. The modulus of each factor is most conveniently obtained graphically (see Ex. 9, Art. 8). A trigonometrical expression for w_1 may be obtained thus:

$$\begin{aligned}(1+b+c)(1+a) &= 1+b+c+a+ab+ac-bc+bc \\ &= 1-bc \\ (a+b+c)(1+a) &= a+b+c+a^2+ab+ac+bc-bc \\ &= a^2-bc.\end{aligned}$$

Hence, if in the expression for w_1 we multiply numerator and denominator by $1 + a$, we get

$$w_1 = \frac{a^2(1-b)(1-c)(1-bc)}{(a-b)(c-a)(a^2-bc)}$$

$$= \frac{a^2bc(b^{-\frac{1}{2}} - b^{\frac{1}{2}})(c^{-\frac{1}{2}} - c^{\frac{1}{2}})(b^{-\frac{1}{2}}c^{-\frac{1}{2}} - b^{\frac{1}{2}}c^{\frac{1}{2}})}{a^2bc(a^{\frac{1}{2}}b^{-\frac{1}{2}} - a^{-\frac{1}{2}}b^{\frac{1}{2}})(c^{\frac{1}{2}}a^{-\frac{1}{2}} - c^{-\frac{1}{2}}a^{\frac{1}{2}})(ab^{-\frac{1}{2}}c^{-\frac{1}{2}} - a^{-1}b^{\frac{1}{2}}c^{\frac{1}{2}})}$$

$$= \frac{\left(e^{\frac{-i\theta_2}{2}} - e^{\frac{i\theta_2}{2}}\right)\left(e^{\frac{-i\theta_3}{2}} - e^{\frac{i\theta_3}{2}}\right)\left(e^{\frac{-i\theta_2}{2}} . e^{\frac{-i\theta_3}{2}} - e^{\frac{i\theta_2}{2}} . e^{\frac{i\theta_3}{2}}\right)}{\left(e^{\frac{i\theta_1}{2}} . e^{\frac{-i\theta_2}{2}} - e^{\frac{-i\theta_1}{2}} . e^{\frac{i\theta_2}{2}}\right)\left(e^{\frac{i\theta_3}{2}} . e^{\frac{-i\theta_1}{2}} - e^{\frac{-i\theta_3}{2}} . e^{\frac{i\theta_1}{2}}\right)\left(e^{i\theta_1} . e^{\frac{-i\theta_2}{2}} . e^{\frac{-i\theta_3}{2}} - e^{-i\theta_1} . e^{\frac{i\theta_2}{2}} . e^{\frac{i\theta_3}{2}}\right.}$$

$$= \frac{\sin \tfrac{1}{2}\theta_2 . \sin \tfrac{1}{2}\theta_3 . \sin \tfrac{1}{2}(\theta_2 + \theta_3)}{\sin \tfrac{1}{2}(\theta_1 - \theta_2) \sin \tfrac{1}{2}(\theta_3 - \theta_1) \sin \tfrac{1}{2}(\theta_2 + \theta_3 - 2\theta_1)} .$$

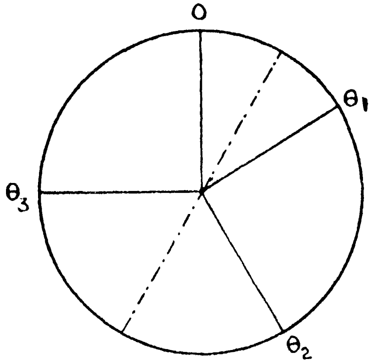

FIG. 37.—Symmetrical four-crank.

The expressions for w_2 and w_3 can be written down from this by permitting the subscripts cyclically.

§ 110. **Symmetrical Four-Crank.**—If the crank angles be symmetrical with respect to the dotted line (Fig. 37), we have $\theta_2 + \theta_3 = 2\pi + \theta_1$.

$$\therefore e^{i(\theta_2 + \theta_3)} = e^{i(2\pi + \theta_1)}$$

$$\therefore e^{i\theta_2} . e^{i\theta_3} = e^{i\theta_1}$$

$$\therefore bc = a.$$

Substituting in the expression for w_1 given in Art. 109, we get

$$\begin{aligned}
w_1 &= \frac{a^2(1-b)(1-c)(1-a)}{a(a-b)(c-a)(a-1)} \\
&= \frac{a(1-b)(1-c)}{(a-b)(a-c)} = \frac{a(1-b-c+bc)}{a^2-ab-ac+bc} \\
&= \frac{a(1-b-c+a)}{a(1-b-c+a)} \\
&= 1.
\end{aligned}$$

Further, we have

$$\begin{aligned}
w_2 &= \frac{b^2(1-c)(1-a)(1+c+a)}{(b-c)(a-b)(a+b+c)} \\
&= \frac{b^2(1-c)(1-a)(1+c+a)}{(b-c)(bc-b)(a+b+c)} \qquad \text{(since } a = bc\text{)} \\
&= \frac{(1-a)(b+bc+ab)}{(b-c)(a+b+c)} \\
&= \frac{(a-1)(a+b+ab)}{(b-c)(a+b+c)} \quad . \quad . \quad . \quad . \quad . \qquad (1)
\end{aligned}$$

Again,

$$\begin{aligned}
w_3 &= \frac{c^2(1-a)(1-b)(1+a+b)}{(c-a)(b-c)(a+b+c)} \\
&= \frac{c^2(1-a)(1-b)(1+a+b)}{(c-bc)(b-c)(a+b+c)} \\
&= \frac{(1-a)(c+ac+bc)}{(b-c)(a+b+c)} = \frac{(1-a)(c+ac+a)}{(b-c)(a+b+c)} \qquad (2) \\
&= \frac{(a-1)(a+b+ab)}{(b-c)(a+b+c)} \qquad \text{(since } c+ac+bc+a+b+ab=0\text{)}
\end{aligned}$$

Therefore we have $w_2 = w_3$.

Hence

$$\begin{aligned}
2w_2 &= w_2 + w_3, \text{ which from (1) and (2) above} \\
&= \frac{(a-1)(a+b+ab)+(1-a)(a+c+ac)}{(b-c)(a+b+c)} \\
&= \frac{a^2-1}{a+b+c} \\
&= \frac{(a^2-1)(a+1)}{(a+b+c)(a+1)} \\
&= \frac{(a^2-1)(a+1)}{a^2-bc} = \frac{(a^2-1)(a+1)}{a^2-a} \\
&= \frac{(a+1)^2}{a} = (a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2 \\
&= \left(e^{\frac{i\theta_1}{2}}+e^{\frac{-i\theta_1}{2}}\right)^2 = 4\cos^2\tfrac{1}{2}\theta_1
\end{aligned}$$

$$\therefore\ w_2 = 2\cos^2\tfrac{1}{2}\theta_1 = 1+\cos\theta_1.$$

In the symmetrical four-crank having primary and secondary balance we therefore have the crank masses equal in pairs: $1 = w_1$; $w_2 = w_3 = 1 + \cos \theta_1$.

§ 111. **Five-Crank Engine with Primary and Secondary Balance.**—The equations to be satisfied are:

$$1 + aw_1 + bw_2 + cw_3 + dw_4 = 0$$

$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 + \frac{1}{d}w_4 = 0$$

$$1 + a^2w_1 + b^2w_2 + c^2w_3 + d^2w_4 = 0$$

$$1 + \frac{1}{a^2}w_1 + \frac{1}{b^2}w_2 + \frac{1}{c^2}w_3 + \frac{1}{d^2}w_4 = 0$$

Solving, we get $w_1 =$

$$\begin{vmatrix} -1 & b & c & d \\ -1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ -1 & b^2 & c^2 & d^2 \\ -1 & \frac{1}{b^2} & \frac{1}{c^2} & \frac{1}{d^2} \end{vmatrix} \div \begin{vmatrix} a & b & c & d \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ a^2 & b^2 & c^2 & d^2 \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} & \frac{1}{d^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & b & c & d \\ 1 & b^3 & c^3 & d^3 \\ 1 & b^4 & c^4 & d^4 \end{vmatrix} \div -\frac{1}{a^2} \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

$$= \frac{a^2(b-1)(c-1)(d-1)(b+c+d+bc+bd+cd)}{(a-b)(d-a)(c-a)(bc+bd+ab+ad+cd+ac)}.$$

This may be evaluated as explained in Art. 109. The expressions for w_2, w_3, w_4 can be written down from the above by permuting the letters cyclically.

CHAPTER XI.

DESIGN OF NON-PLANAR CRANK SYSTEMS WITH SPECIFIED BALANCE.

§ 112. **Introductory.**—Let the cylinders of a proposed engine be pitched at distances $l_0, l_1, l_2, \ldots$ from a fixed reference plane taken normal to the crankshaft. We may well take the plane of the flywheel as our reference plane. The inertia forces of the different cylinders, acting along the cylinder centre lines, now form a system of parallel forces. It is therefore possible that such a system reduce to a pitching couple on the engine frame when, if considered as a concurrent system, it would be in equilibrium. For example, take a two-crank at 180° cylinder-in-line engine (Fig. 38). The primary inertia-loads due to the reciprocating crank masses, shown by the arrows, would give equilibrium if the cranks were coplanar. It is obvious, however, that they now form a pitching couple of moment Fa on the engine. For the couple balance of the general case of the cylinder in line engine, we consider the equilibrium of the crankshaft (or engine-frame) under the action of the inertia loads, as if it were a beam correspondingly loaded.

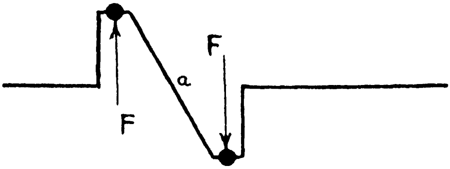

Fig. 38.—Pitching couple of cranks at 180°.

§ 113. **Equations for Balance of Couples Occasioned by nth Harmonic.**—In the engine (Fig. 39) the inertia loads (nth harmonic) are, with the notation of Art. 103 :—

$$p_n\omega^2 m_0 \cos na\,;\ p_n\omega^2 m_1 \cos n(a+\theta_1)\,;\ p_n\omega^2 m_2 \cos n(a+\theta_2)\ \ldots$$

These forces give a couple about a horizontal diameter of the flywheel, which is

$$C_n = p_n\omega^2\{l_0m_0\cos n\alpha + l_1m_1\cos n(\alpha+\theta_1) + l_2m_2\cos n(\alpha+\theta_2) + \ldots\}$$

$$C_n = \cdot 5p_n\omega^2[l_0m_0\{e^{ni\alpha} + e^{-ni\alpha}\} + l_1m_1\{e^{ni(\alpha+\theta_1)} + e^{-ni(\alpha+\theta_1)}\} + \ldots].$$

$$= \cdot 5p_n\omega^2 e^{ni\alpha}\{l_0m_0 + l_1m_1e^{ni\theta_1} + l_2m_2e^{ni\theta_2} + \ldots + e^{-2ni\alpha}(l_0m_0 + l_1m_1e^{-ni\theta_1} + l_2m_2e^{-ni\theta_2} + \ldots)\}.$$

To have this couple zero for all values of α, we evidently must have both the series in circumflex brackets separately zero, i.e.,

$$l_0m_0 + l_1m_1e^{ni\theta_1} + l_2m_2e^{ni\theta_2} + \ldots = 0 \quad . \quad . \quad (1)$$

$$l_0m_0 + l_1m_1e^{-ni\theta_1} + l_2m_2e^{-ni\theta_2} + \ldots = 0 \quad . \quad (2)$$

In this chapter it is understood that we have already provided for the balance of the engine considered as a coplanar system, so that we will have the equations of Art. 103 satisfied. Multiplying equation (1A) of Art. 103 by l_0, we get

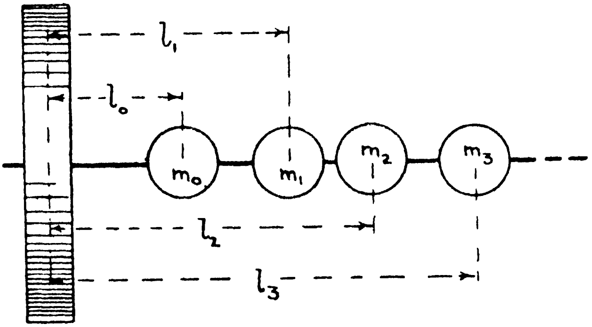

FIG. 39.—Effect of cylinder pitch on balance of pitching couples.

$$l_0m_0 + l_0m_1e^{ni\theta_1} + l_0m_2e^{ni\theta_2} + \ldots = 0.$$

Subtracting this from (1) above gives

$$(l_1 - l_0)m_1e^{ni\theta_1} + (l_2 - l_0)m_2e^{ni\theta_2} + \ldots = 0.$$

which we may write

$$p_1m_1e^{ni\theta_1} + p_2m_2e^{ni\theta_2} + \ldots = 0 \quad . \quad . \quad (3)$$

It will be evident that p_1, p_2, . . . are the distances of the respective cylinders from cylinder No. 0.

Again multiplying (2A) of Art. 103 by l_0 and subtracting from (2) above, we obtain on reduction

$$p_1m_1e^{-ni\theta_1} + p_2m_2e^{-ni\theta_2} + \ldots = 0 \quad . \quad . \quad (4)$$

Dividing (3) and (4) by $p_1 m_o$, and writing y_2 for p_2/p_1, y_3 for p_3/p_1, etc., we get

$$w_1 e^{ni\theta_1} + y_2 w_2 e^{ni\theta_2} + y_3 w_3 e^{ni\theta_3} + \ldots = 0 \quad . \quad . \quad (5)$$
$$w_1 e^{-ni\theta_1} + y_2 w_2 e^{-ni\theta_2} + y_3 w_3 e^{-ni\theta_3} + \ldots = 0 \quad . \quad (6)$$

It will be evident that $y_2, y_3, \ldots$ are the distances of the respective cylinders from cylinder No. 0, taking the distance of cylinder No. 1 as unit. When equations (5), (6) are satisfied, we know that the nth harmonic of the inertia forces do not form a couple.

§ 114. **Equations for Balance of Primary and Secondary Couples.**—Putting $n = 1$ in the equations (5), (6) of Art. 113, we get the conditions for balance of the primary inertia couple to be

$$w_1 e^{i\theta_1} + y_2 w_2 e^{i\theta_2} + y_3 w_3 e^{i\theta_3} + \ldots = 0 \quad . \quad . \quad (7)$$
$$w_1 e^{-i\theta_1} + y_2 w_2 e^{-i\theta_2} + y_3 w_2 e^{-i\theta_3} + \ldots = 0 \quad . \quad (8)$$

Again, putting $n = 2$ in (5), (6), we get the conditions that the secondary inertia forces do not form a couple to be

$$w_1 e^{2i\theta_1} + y_2 w_2 e^{2i\theta_2} + y_3 w_3 e^{2i\theta_3} + \ldots = 0 \quad . \quad . \quad (9)$$
$$w_1 e^{-2i\theta_1} + y_2 w_2 e^{-2i\theta_2} + y_3 w_3 e^{-2i\theta_3} + \ldots = 0 \quad . \quad (10)$$

For balance of both primary and secondary couples, equations (7), (8), (9), (10) must be satisfied. We therefore get eight equations to represent the conditions for balance of primary and secondary forces and couples, namely, (3), (4), (5), (6) of Art. 103, and (7), (8), (9), (10) of this Art. Writing a for $e^{i\theta_1}$; b for $e^{i\theta_2}$, etc., these eight equations are

$$1 + aw_1 + bw_2 + cw_3 + \ldots = 0 \quad . \quad . \quad (11)$$
$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 + \ldots = 0 \quad . \quad . \quad (12)$$
$$1 + a^2 w_1 + b^2 w_2 + c^2 w_3 + \ldots = 0 \quad . \quad . \quad (13)$$
$$1 + \frac{1}{a^2}w_1 + \frac{1}{b^2}w_2 + \frac{1}{c^2}w_3 + \ldots = 0 \quad . \quad . \quad (14)$$
$$aw_1 + by_2 w_2 + cy_3 w_3 + \ldots = 0 \quad . \quad (15)$$
$$\frac{1}{a}w_1 + \frac{1}{b}y_2 w_2 + \frac{1}{c}y_3 w_3 + \ldots = 0 \quad . \quad (16)$$
$$a^2 w_1 + b^2 y_2 w_2 + c^2 y_3 w_3 + \ldots = 0 \quad . \quad (17)$$
$$\frac{1}{a^2}w_1 + \frac{1}{b^2}y_2 w_2 + \frac{1}{c^2}y_3 w_3 + \ldots = 0 \quad . \quad (18)$$

An examination of these eight equations will reveal that for an n crank engine, there are $3n - 4$ variables: we have $(n - 1)$ variables representing angles, $(n - 1)$ w's, and $(n - 2)$ y's, giving a total $3n - 4$. For the complete primary and secondary balance of an engine, eight of these variables must be left unassigned, and be determined from the solution of the above eight simultaneous equations. The least number of (non-planar) cranks with which it is possible to satisfy the eight equations is therefore given by writing

$$3n - 4 = 8, \text{ or } n = 4.$$

A four-crank engine is the minimum size we can arrange for complete balance of primary and secondary inertia loads. It also follows that the complete primary and secondary balance of any crank system can be secured by the addition of three extra cranks.

§ 115. **Development from Coplanar Systems.**—From the coplanar crank systems found in Chapter X. we can derive non-planar systems with equally good balance, both as regards inertia forces and inertia couples. Consider the two crank coplanar crank system balanced for primary forces (Fig. 40.) Since a reciprocating mass w is dynamically equivalent to two detached reciprocating masses m_1 and m_2, provided $m_1 + m_2 = w$ and also that m_1 with m_2 has the same centre of mass as w, we may replace the reciprocating mass w (Fig. 40) by the reciprocating masses m_1, m_2, and operate them by separate cranks without disturbing the balance. Thus we obtain a three-crank system (Fig. 40), which is in balance for primary forces and couples provided

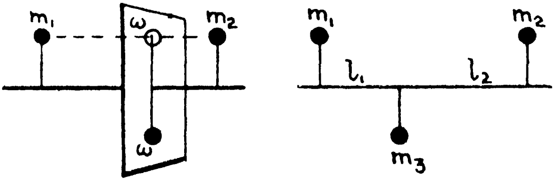

Fig. 40.—Three-crank cylinder in line engine with balance of primary forces and couples.

$$m_1 l_1 = m_2 l_2$$
$$m_1 + m_2 = m_3.$$

By an obvious extension of this method we can, from the primary balanced coplanar two-crank, obtain a four-crank cylinder in line engine in balance for primary forces and couples. The equal crank masses w_1, w_2 of the coplanar system (Fig 41) are each broken into two detached reciprocating masses m_1, m_2, and m_3, m_4. We shall have the balance of the coplanar type unaltered provided

$$m_1 + m_2 = m_3 + m_4 = w$$
$$m_1 l_1 = m_2 l_2$$
$$m_3 l_3 = m_4 l_4.$$

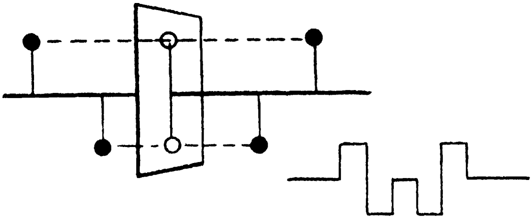

FIG. 41.—Four-crank cylinder in line engine with balance of primary forces and couples.

Putting $m_1 = m_2 = m_3 = m_4$ we get the usual design for the crankshaft of the four-cylinder petrol engine, shown on right (Fig. 41).

From the coplanar three-crank at 120° of Art. 106, in balance for primary and secondary forces, we can derive a cylinder in line type with an equally good balance. Fig. 42 shows a four-crank design having primary and secondary force and couple balance. Here one of the coplanar cranks is broken into two cranks with masses m_1, m_2. We must have $m_1 + m_2 = w$ and $m_1 l_1 = m_2 l_2$. Fig. 43 shows a primary and secondary

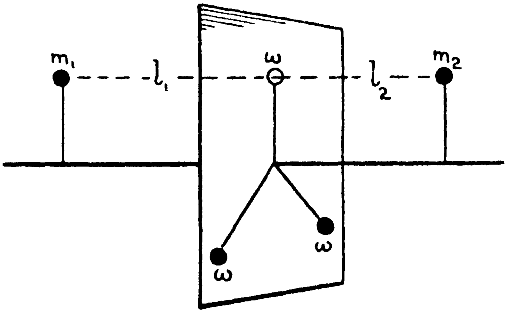

FIG. 42.—Four-crank cylinder in line engine with balance of primary and secondary inertia forces and pitching couples.

balanced five-crank engine derived from the coplanar three-crank, while Fig. 44 shows the usual design for the crankshaft of the six-cylinder petrol engine evolved from the coplanar three-crank at 120°.

From the coplanar four-crank designed for primary and secondary balance in Art. 107 we can, by the method of duplicate cranks above illustrated, design equally well-balanced crank shafts for cylinder in line engines. For example, from the four-crank at 90° we get three designs (Fig. 45), for the eight-crank cylinder in line having primary and secondary force and couple balance. In each of the designs it is obvious that if the pitches be symmetrical and the crank masses equal, the centre of mass of each parallel pair of cranks is at the unfilled circle in the central plane. The lay-out first shown is that adopted by the Leyland motors. The Isotta Fraschini (Fig. 46) is a lay-out which has the disadvantage of giving a secondary pitching couple on the engine.

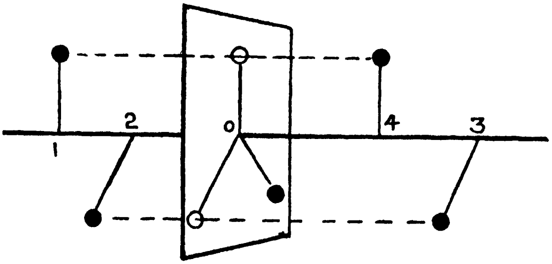

FIG. 43.—Five-crank with primary and secondary balance of inertia forces and pitching couples $m_1 + m_4 = m_O = m_2 + m_3$
$m_1 l_1 = m_4 l_4$; $m_2 l_2 = m_3 l_3$.

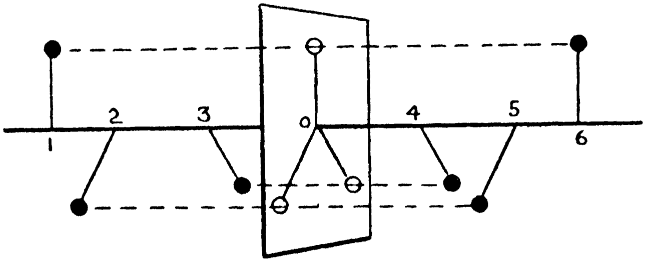

FIG. 44.—Six-crank with primary and secondary balance of forces and pitching couples. All reciprocating masses equal and pitches symmetrical about central plane.

§ 116. **Application of the General Equations. Three-Crank Engine.**—Writing down the equations for primary balance from Art. 114, we get for a three-crank engine:

$$1 + aw_1 + bw_2 = 0 \quad . \quad . \quad . \quad (1)$$

$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 = 0 \quad . \quad . \quad . \quad (2)$$

$$aw_1 + by_2w_2 = 0 \quad . \quad . \quad . \quad (3)$$

$$\frac{1}{a}w_1 + \frac{1}{b}y_2w_2 = 0 \quad . \quad . \quad . \quad (4)$$

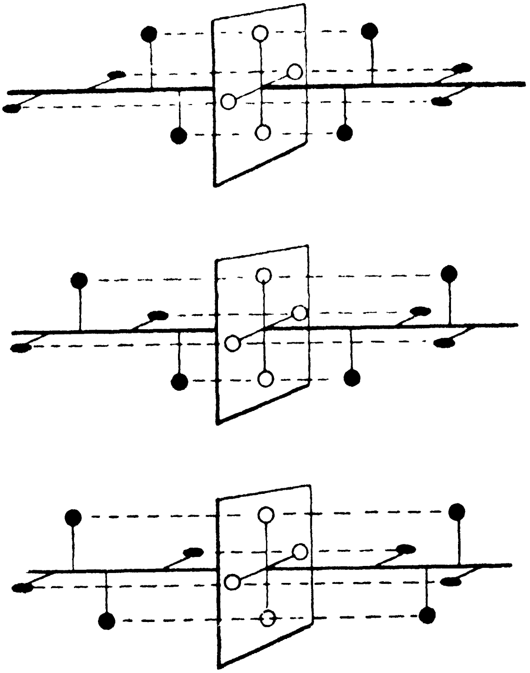

Fig. 45.—Three designs for eight-crank cylinder in line engine with complete primary and secondary balance.

If we assign the value of y_2, the remaining variables a, b, w_1, w_2 may evidently be determined from the four equations. The product of (3) and (4) gives

$$w_1 = + y_2w_2 \quad . \quad . \quad . \quad (5)$$

If y be $\pm$ we must take the $\pm$ value to have w_1 positive. Subtracting (3) from (1), and (4) from (2), we get

$$1 + bw_2(1 - y) = 0 \quad . \quad . \quad . \quad (5\text{A})$$

$$1 + \frac{1}{b} w_2(1 - y) = 0 \quad . \quad . \quad . \quad (5\text{B})$$

From these we get at once (taking products)

$$1 = \pm w_2(1 - y) \quad . \quad . \quad . \quad (6)$$

If $y > 1$ we must take the $-$ value to have w_2 positive, while if $y < 1$ (or negative) we must take the $+$ value to have w_2 positive. We shall examine these cases separately.

Case I.—$y > 1$. We have from (6)

$$1 = -w_2(1 - y) = -w_2 + yw_2$$

$$w_1 = yw_2 \quad . \quad . \quad . \quad (\text{from } 5)$$

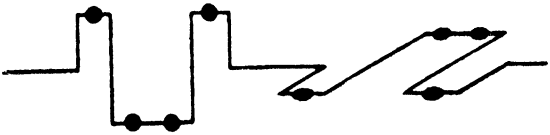

FIG. 46.—Eight-crank with unbalanced secondary pitching couple.

Hence

$$1 = -w_2 + w_1, \text{ or } 1 + w_2 = w_1$$

$$w_2 = \frac{1}{y - 1} \quad . \quad . \quad . \quad (\text{from } 6)$$

Hence, substituting in (5), we have

$$w_1 = \frac{y}{y - 1}.$$

Substituting for w_2 in (5A), we have

$$1 + b\frac{1}{y - 1}(1 - y) = 0.$$

Therefore $b = 1$, that is, $e^{i\theta_2} = 1$, so that $\theta_2 = 2\pi$, and this crank is consequently parallel to crank No. 0. Putting this value of b in (1), we get

$$1 + aw_1 + w_2 = 0,$$

and since $1 + w_2 = w_1$, this gives $a = -1$, that is, $e^{i\theta_1} = -1$, or $\theta_1 = 180°$. The crank system is therefore as shown in Fig. 47.

Case II.—$y < 1$. Equation (6) becomes

$$1 = w_2 - yw_2, \text{ or } w_2 = \frac{1}{1 - y}.$$

Substituting $w_1 = y_2 w_2$ from (5), we get

$$1 = w_2 - w_1, \text{ and } w_1 = \frac{y}{1 - y}.$$

Now, substituting for w_2 in (5A), we obtain

$$1 + b \,.\, \frac{1}{1 - y}(1 - y) = 0.$$

$$\therefore b = -1.$$

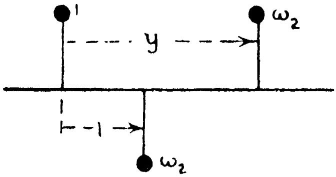

FIG. 47.—Three-crank cylinder in line engine with primary force and couple balance, $w_1 = 1 + w_2$; $w_2y = w_1$.

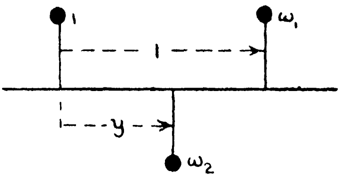

FIG. 48.—Three-crank cylinder in line engine with primary force and couple balance, $w_2 = 1 + w_1$; $w_2y = w_1$.

Putting this value of b in (1), we get

$$1 + aw_1 - w_2 = 0$$
$$\therefore aw_1 = w_2 - 1 = w_1.$$
$$\therefore a = 1.$$

Hence we have $\theta_1 = 2\pi$; $\theta_2 = \pi$. This crank system is shown in Fig. 48.

Case III.—y negative. As in Case II, we get

$$w_2 = \frac{1}{1 - y}$$

while substituting in (5) we have

$$w_1 = \frac{-y}{1 - y}.$$

Hence we have $w_1 + w_2 = 1$. Substituting for w_2 in (5A), we get $b = -1$, and putting this value of b in (1), we get

$$1 + aw_1 - w_2 = 0$$
$$\therefore w_1 + w_2 + aw_1 - w_2 = 0.$$
$$\therefore a = -1.$$

Hence $\theta_1 = \pi = \theta_2$. This crank system is shown in Fig. 49.

§ 117. **Four-Crank Engine with Primary Force and Couple Balance.**—With four cranks the equations for primary force and couple balance (Art. 114) are

$$1 + aw_1 + bw_2 + cw_3 = 0 \quad . \quad . \quad . \quad (1)$$
$$1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 = 0 \quad . \quad . \quad . \quad (2)$$
$$aw_1 + by_2w_2 + cy_3w_3 = 0 \quad . \quad . \quad . \quad (3)$$
$$\frac{1}{a}w_1 + \frac{1}{b}y_2w_2 + \frac{1}{c}y_3w_3 = 0 \quad . \quad . \quad . \quad (4)$$

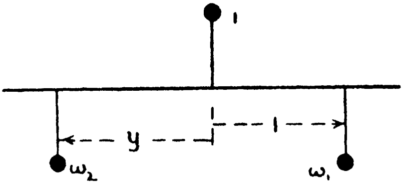

FIG. 49.—Three-crank cylinder in line engine with primary force and couple balance $1 = w_2 + w_1$; $w_2y = w_1$.

Subtracting (3) from (1) and (4) from (2), we get

$$bw_2(1 - y_2) + cw_3(1 - y_3) = -1$$
$$\frac{1}{b}w_2(1 - y_2) + \frac{1}{c}w_3(1 - y_3) = -1.$$

Equating the product of the left-hand members to the product of the right-hand members, we get

$$w_2^2(1 - y_2)^2 + w_3^2(1 - y_3)^2 + w_2w_3(1 - y_2)(1 - y_3)(c/b + b/c) = 1 \quad (5)$$

Equations (3) and (4) may be written

$$-aw_1 = by_2w_2 + cy_3w_3$$
$$-\frac{1}{a}w_1 = \frac{1}{b}y_2w_2 + \frac{1}{c}y_3w_3.$$

By equating products of corresponding sides, we obtain

$$w_1^2 = y_2^2 w_2^2 + y_3^2 w_3^2 + w_2 w_3 y_2 y_3 \left(\frac{c}{b} + \frac{b}{c}\right).$$

$$\therefore\ w_2 w_3 \left(\frac{c}{b} + \frac{b}{c}\right) = \frac{w_1^2 - y_2^2 w_2^2 - y_3^2 w_3^2}{y_2 y_3}.$$

Substituting this value in (5), we have

$$w_1^2(1-y_2)^2 + w_3^2(1-y_3)^2 + \frac{(1-y_2)(1-y_3)(w_1^2 - y_2^2 w_2^2 - y_3^2 w_3^2)}{y_2 y_3} = 1$$

which on reduction gives the following relation between the cylinder pitches and the crank masses:

$$(1-y_2)(y_3-y_2)y_2 w_2^2 + (1-y_3)(y_2-y_3)y_3 w_3^2 + (1-y_2)(1-y_3)w_1^2 = y_2 y_3 \quad . \quad (6)$$

§ 118. **Four-Crank with Symmetrical Cylinder Pitches.**—If the cylinders be spaced symmetrically as in Fig. 50, so that

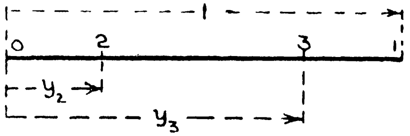

FIG. 50.—Symmetrically pitched four-crank with primary force and couple balance.

$y_2 = 1 - y_3$, we obtain on substitution in (6) of Art. 117:

$$(1-y_2)(1-2y_2)y_2 w_2^2 + y_2(2y_2-1)(1-y_2)w_3^2 + (1-y_2)y_2 w_1^2 = y_2(1-y_2)$$

$$\therefore\ (1-2y_2)w_2^2 + (2y_2-1)w_3^2 + w_1^2 = 1$$

$$\therefore\ 1 - w_1^2 = (1-2y_2)(w_2^2 - w_3^2).$$

It will be readily seen that $1 - 2y_2$ is the distance between the centre pair of cylinders (the distance between the outer pair being taken as unity). Calling this distance l, we have the simple relation,

$$1 - w_1^2 = l(w_2^2 - w_3^2).$$

§ 119. **Determination of Cylinder Pitches given Crank Angles and Reciprocating Masses.**—Equations (3) and (4) of Art. 117 are

$$a w_1 + b w_2 y_2 + c w_3 y_3 = 0 \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{a} w_1 + \frac{1}{b} w_2 y_2 + \frac{1}{c} w_3 y_3 = 0 \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by $\frac{1}{b}$ and (2) by b, we get

$$\frac{a}{b}w_1 + y_2w_2 + \frac{c}{b}y_3w_3 = 0$$

$$\frac{b}{a}w_1 + y_2w_2 + \frac{b}{c}y_3w_3 = 0$$

By subtraction we get

$$w_1\left(\frac{a}{b} - \frac{b}{a}\right) + y_3w_3\left(\frac{c}{b} - \frac{b}{c}\right) = 0.$$

$$\therefore y_3 = \frac{w_1}{w_3} \cdot \frac{\frac{a}{b} - \frac{b}{a}}{\frac{b}{c} - \frac{c}{b}}$$

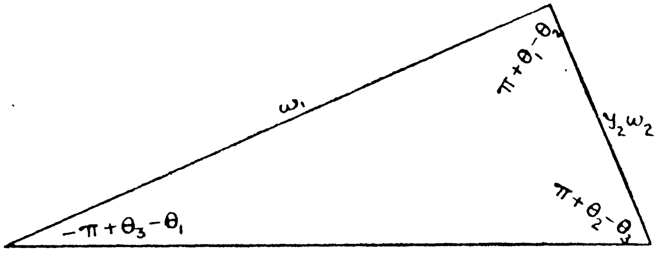

FIG. 51.—Law of Sines for four-crank with primary force and couple balance.

which, on putting in the equivalent exponential values, becomes

$$y_3 = \frac{w_1}{w_3}\left\{\frac{e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)}}{e^{i(\theta_2-\theta_3)} - e^{-i(\theta_2-\theta_3)}}\right\}$$

$$= \frac{w_1}{w_3}\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_2-\theta_3)}.$$

In like manner, by multiplying (1) by $\frac{1}{c}$ and (2) by c, we obtain

$$y_2 = \frac{w_1}{w_2}\frac{\sin(\theta_3-\theta_1)}{\sin(\theta_2-\theta_3)}.$$

§ 120. **Geometrical Relations.**—Referring to Fig. 51, the relations found in Art. 119 will be recognised as " The Law of Sines " for the triangle we have drawn.

Graphical solutions of problems dependent upon the above relation will, therefore, at once suggest themselves.

If for the crank system 0, 1, 2, 3 (Fig. 52), having primary forces balanced, we draw end to end the vectors 1, $w_1 e^{i\theta_1}$, $w_2 e^{i\theta_2}$, $w_3 e^{i\theta_3}$, we obtain the closed figure ABCD, since $1 + w_1 e^{i\theta_1} + w_2 e^{i\theta_2} + w_3 e^{i\theta_3} = 0$.

If now through B we draw BE parallel to AD, it is obvious that the triangle BCE is identical with the triangle of Fig. 51. Hence CE/CD $= y_2$. Again, if through B we draw BF parallel to CD, we get FD = BE $= y_3 w_3$. Hence FD/AD $= y_3$.

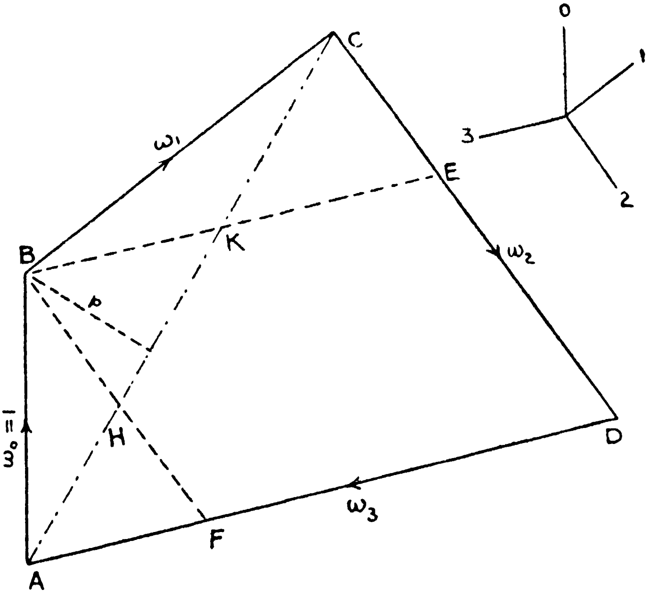

FIG. 52.—Relation between crank angles and cylinder pitches of four-crank engine with primary force and couple balance.

A very interesting relation now discloses itself on joining AC. Let H and K be the crosses of AC with BF and BE. In the triangle ACD, since KE is parallel to AD, we get CE/CD = CK/AC.

$$\therefore \text{CK/AC} = y_2.$$

Similarly, since FH is parallel to CD,

$$\text{FD/AD} = \text{CH/AC} = y_3.$$

Hence

$$\text{CK} : \text{CH} : \text{AC} :: y_2 : y_3 : 1,$$

that is, if AC represent 1, the distances CK and CH represent y_2 and y_3 respectively.

We may note that if the crank angles are symmetrical and $w_1 = 1$, $w_2 = w_3$, as they are for the engine of Art. 110, then Fig. 52 is symmetrical with respect to BD, and hence the cylinder pitches are symmetrical.

§ 121. **Relation between the Crank Angles and the Cylinder Pitches.**—For primary force and couple balance of the four-crank engine, we require equations (11), (12), (15), (16) of Art. 114 satisfied. These equations are re-written here:

$$\begin{aligned} 1 + aw_1 + bw_2 + cw_3 &= 0 \\ 1 + \frac{1}{a}w_1 + \frac{1}{b}w_2 + \frac{1}{c}w_3 &= 0 \\ 0 + aw_1 + by_2w_2 + cy_3w_3 &= 0 \\ 0 + \frac{1}{a}w_1 + \frac{1}{b}y_2w_2 + \frac{1}{c}y_3w_3 &= 0. \end{aligned}$$

Eliminating the w's we get

$$\begin{vmatrix} 1 & a & b & c \\ 1 & \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ 0 & a & by_2 & cy_3 \\ 0 & \frac{1}{a} & \frac{y_2}{b} & \frac{y_3}{c} \end{vmatrix} = 0.$$

$$\therefore \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a^2 & b^2 & c^2 \\ 0 & a^2 & b^2y_2 & c^2y_3 \\ 0 & 1 & y_2 & y_3 \end{vmatrix} = 0.$$

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ a^2 & b^2y_2 & c^2y_3 \\ 1 & y_2 & y_3 \end{vmatrix} - 1 \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & b^2y_2 & c^2y_3 \\ 0 & y_2 & y_3 \end{vmatrix} + 1 \begin{vmatrix} 1 & a^2 & c^2 \\ 0 & a^2 & c^2y_3 \\ 0 & 1 & y_3 \end{vmatrix} - 1 \begin{vmatrix} 1 & a^2 & b^2 \\ 0 & a^2 & b^2y_2 \\ 0 & 1 & y_2 \end{vmatrix} = 0.$$

$$\begin{aligned} \therefore\ a^2b^2y_2y_3 + b^2c^2y_3 + c^2a^2y_2 - b^2c^2y_2 - c^2a^2y_2y_3 - a^2b^2y_3 \\ - b^2y_2y_3 + c^2y_2y_3 + a^2y_3 - c^2y_3 - a^2y_2 + b^2y_2 = 0. \end{aligned}$$

$$\therefore (y_2 - y_2y_3)(c^2 - 1)(a^2 - b^2) + (y_3 - y_2y_3)(b^2 - 1)(c^2 - a^2) = 0$$

$$\therefore -\frac{y_2(1 - y_3)}{y_3(1 - y_2)} = \frac{(b^2 - 1)(c^2 - a^2)}{(c^2 - 1)(a^2 - b^2)} = \frac{\left(b - \frac{1}{b}\right)\left(\frac{c}{a} - \frac{a}{c}\right)}{\left(c - \frac{1}{c}\right)\left(\frac{a}{b} - \frac{b}{a}\right)} ;$$

$$= \frac{(e^{i\theta_2} - e^{-i\theta_2})(e^{i\theta_3} . e^{-i\theta_1} - e^{-i\theta_3} . e^{i\theta_1})}{(e^{i\theta_3} - e^{-i\theta_3})(e^{i\theta_1} . e^{-i\theta_2} - e^{-i\theta_1} . e^{i\theta_2})}$$

$$= \frac{\sin\theta_2 . \sin(\theta_3 - \theta_1)}{\sin\theta_3 . \sin(\theta_1 - \theta_2)}$$

$$\therefore \frac{(1 - y_3)\sin\theta_3}{(1 - y_2)\sin\theta_2} = \frac{y_3\sin(\theta_1 - \theta_3)}{y_2\sin(\theta_1 - \theta_2)}.$$

It is interesting to deduce this relation from the geometry of Fig. 52. We have there:

$$\begin{aligned} CK . p &= BC . BK \sin CBK \\ CH . p &= BC . BH \sin CBH \\ AK . p &= AB . BK . \sin ABK \\ AH . p &= AB . BH \sin ABH. \end{aligned}$$

$$\therefore \frac{CK}{CH} = \frac{BK \sin CBK}{BH \sin CBH}.$$

$$\frac{AH}{AK} = \frac{BH \sin ABH}{BK \sin ABK}.$$

$$\therefore \frac{CK . AH}{CH . AK} = \frac{\sin CBK . \sin ABH}{\sin CBH . \sin ABK}.$$

$$\therefore \frac{CK(CA - CH)}{CH(CA - CK)} = \frac{\sin(\theta_3 - \theta_1)\sin\theta_2}{\sin(\theta_1 - \theta_2)\sin\theta_3}.$$

§ 122. **Determination of Reciprocating Masses: Special Engine.**—The engine here discussed is the four-crank having the angles of a symmetrical four-crank with primary and secondary balance. For such an engine we have, from Art. 110, $w_2 = w_3$; $w_1 = 1$; $b + c = -2a/(1 + a)$; $a = bc$. Equation (11), Art. 114, now becomes

$$1 + a + bw_2 + cw_2 = 0.$$

$$\therefore w_2 = -\frac{1 + a}{b + c}.$$

Equation (15), Art. 114, becomes

$$a + by_2w_2 + cy_3w_2 = 0.$$

From Fig. 53 it is evident that we can put $y_2 = \frac{1}{2}(1 - x)$,

$y_3 = \frac{1}{2}(1 + x)$, and setting these values in the foregoing equation, we have

$$a + w_2\{\tfrac{1}{2}(b - bx) + \tfrac{1}{2}(c + cx)\} = 0$$
$$2a + w_2(b + c) + w_2x(c - b) = 0$$
$$\therefore 2a + -(1 + a) + w_2x(c - b) = 0.$$

$$\therefore w_2 = \frac{a - 1}{x(b - c)}.$$

$$\therefore x^2w_2^2 = \frac{(a - 1)^2}{(b - c)^2} = \frac{(1 + a)^2 - 4a}{(b + c)^2 - 4bc}$$

$$= \frac{(1 + a)^2 - 4a}{\dfrac{4a^2}{(1 + a)^2} - 4a} = \frac{\dfrac{(1 + a)^2}{2a} - 2}{\dfrac{2a}{(1 + a)^2} - 2}$$

$$= \frac{w_2 - 2}{\dfrac{1}{w_2} - 2}.$$

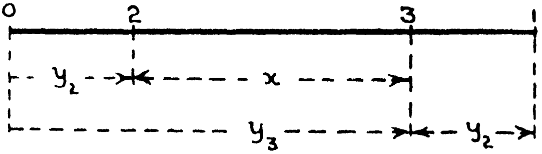

FIG. 53.—Symmetric four-crank with primary and secondary force balance and primary couple balance.

$\therefore 2x^2w_2^2 + w_2(1 - x^2) - 2 = 0$, from which w_2 can be determined when x is given. The design can now be completed by reference to Art. 110.

§ 123. **Engine with Odd Number of Cylinders Symmetrically Pitched About Centre Cylinder; Angles, Masses, and Crank Radii all Equal.**—Let the central cylinder be No. 0. Putting all the crank masses equal, and writing $\theta_1 = \lambda$, $\theta_2 = 2\lambda$, $\theta_3 = 3\lambda$, etc., in the general equations for balance of the nth order inertia couple, (5), (6) of Art. 113 become when the number of cylinders is $k + 1$.

$$e^{ni\lambda} + y_2e^{2ni\lambda} + y_3e^{3ni\lambda} + \ldots + y_ke^{kni\lambda} = 0 \quad . \quad . \quad (1)$$
$$e^{-ni\lambda} + y_2e^{-2ni\lambda} + y_3e^{-3ni\lambda} + \ldots + y_ke^{-kni\lambda} = 0 \quad (2)$$

Multiplication of (1) by $e^{-(k+1)ni\lambda}$ gives

$$e^{-kni\lambda} + y_2e^{-(k-1)ni\lambda} + \ldots + y_{k-1}e^{-2ni\lambda} + y_ke^{-ni\lambda} = 0$$

Therefore, if in (1) and (2) we put

$$\begin{aligned} y_k &= -1 \\ y_2 &= -y_{k-1} \\ y_3 &= -y_{k-2} \\ \dots &= \text{etc.} \end{aligned}$$

we get

$$e^{ni\lambda} + y_2 e^{2ni\lambda} + y_3 e^{3ni\lambda} + \dots - y_2 e^{(k-1)ni\lambda} - e^{kni\lambda} = 0 \quad (3)$$
$$e^{-ni\lambda} + y_2 e^{-2ni\lambda} + y_3 e^{-3ni\lambda} + \dots - y_2 e^{-(k-1)ni\lambda} - e^{-kni\lambda} = 0 \quad (4)$$

Since equation (4) follows from (3) by multiplying it by $e^{-(k+1)ni\lambda}$ it will be sufficient for balance of the nth order inertia couple if equation (3) be satisfied. Dividing (3) by $e^{ni\lambda}$ there results

$$1 + y_2 e^{ni\lambda} + y_3 e^{2ni\lambda} + \dots - y_2 e^{(k-2)ni\lambda} - e^{(k-1)ni\lambda} = 0$$

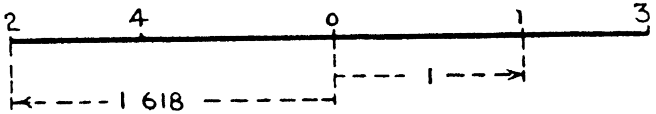

FIG. 54.—Cylinder pitches of five-crank with primary pitching couple balanced.

which may be written

$$(1 - e^{(k-1)ni\lambda}) + y_2(e^{ni\lambda} - e^{(k-2)ni\lambda}) + y_3(e^{2ni\lambda} - e^{(k-3)ni\lambda})$$
$$+ \dots + y_{\frac{1}{2}k}(e^{(\frac{1}{2}k-1)ni\lambda} - e^{-\frac{1}{2}kni\lambda}).$$

Dividing throughout by $-e^{\frac{1}{2}(k-1)ni\lambda}$, this becomes

$$\{e^{\frac{1}{2}(k-1)ni\lambda} - e^{-\frac{1}{2}(k-1)ni\lambda}\} + y_2\{e^{\frac{1}{2}(k-3)ni\lambda} - e^{-\frac{1}{2}(k-3)ni\lambda}\}$$
$$+ y_3\{e^{\frac{1}{2}(k-5)ni\lambda} - e^{-\frac{1}{2}(k-5)ni\lambda}\} + \dots + y_{\frac{1}{2}k}\{e^{\frac{1}{2}ni\lambda} - e^{-\frac{1}{2}ni\lambda}\} = 0.$$

$$\therefore \sin \tfrac{1}{2}(k-1)n\lambda + y_2 \sin \tfrac{1}{2}(k-3)n\lambda + y_3 \sin \tfrac{1}{2}(k-5)n\lambda$$
$$+ \dots + y_{\frac{1}{2}k} \sin \tfrac{1}{2}n\lambda = 0.$$

§ 124. **Application to Five-crank at 72°.**—The foregoing equation becomes for $n = 1$, and $k + 1 = 5$, $\lambda = 72°$:

$$\sin \frac{3\lambda}{2} + y_2 \sin \frac{\lambda}{2} = 0$$
$$\sin 108° + y_2 \sin 36° = 0$$

from which $-y_2 = \sin 108°/\sin 36° = \sin 72°/\sin 36° = 2 \sin 36° \cos 36°/\sin 36° = 2 \cos 36° = 1{\cdot}618$.

The negative sign means that cylinders No. 2 and No. 1 are on the opposite sides of No. 0. The positions of cylinders No. 3 and No. 4 are given by symmetry. The crank settings from left to right (Fig. 54) are 144°, 288°, 0°, 72°, 216°. Reference to Art. 18 will show that this engine is in balance for primary and secondary forces. With the above cylinder pitches, it is in balance for primary couples.

§ 125. **Application to Seven-crank at 51° 26′.**—Substituting in the equation of Art. 123, we get for balance of primary couples:

$$\sin \frac{5}{2} . \lambda + y_2 \sin \frac{3}{2}\lambda + y_3 \sin \frac{\lambda}{2} = 0.$$

$$\cdot 7819 + \cdot 9749 y_2 + \cdot 4339 y_3 = 0 \quad . \quad . \quad . \quad (1)$$

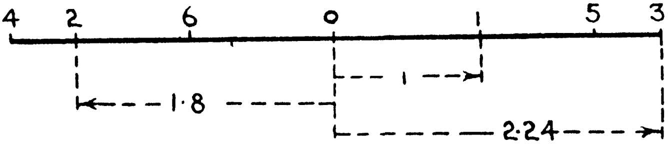

FIG. 55.—Cylinder pitch of seven-crank with primary and secondary pitching couples balanced.

Putting $y_3 = -\frac{1}{3}$, we get $y_2 \doteqdot -\frac{2}{3}$.

Hence we have the interesting result that the seven cylinders arranged with cranks in the sequence 6λ, 2λ, 3λ, 0, 4λ, 5λ, λ; and at practically equal pitches are in balance for primary couples, as well as being in balance for primary and secondary forces. For balance of the secondary couples we require:—

$$\sin 5\lambda + y_2 \sin 3\lambda + y_3 \sin \lambda = 0$$

$$\therefore - \cdot 9749 + \cdot 4339 y_2 + \cdot 7819 y_3 = 0.$$

Combining this with the equation (1) above and solving, we find y_2 and y_3, which will give the engine in complete primary and secondary balance. The solution gives $y_2 = -1{\cdot}8$, $y_3 = 2{\cdot}24$, and this result is set out in Fig. 55.

APPENDIX.

EXPANSION OF FUNCTIONS IN TRIGONOMETRIC SERIES.

Fourier Series.—It is convenient to give here the expansions in trigonometric or Fourier Series of certain functions which arise out of the kinematics of the slider crank chain mechanism. In each case the possibility of the expansion is assumed. Further, it is assumed that in the case of the differentiated series discussed, the derivative of the function is given by the limit of the sum of the series obtained by differentiating its expansion term by term. All the expansions are rapidly convergent, ρ being taken less than one.

1.—Expand $f(\theta) = \dfrac{1 - \rho\cos\theta}{1 - 2\rho\cos\theta + \rho^2}$.

We have

$$\frac{2 - 2\rho\cos\theta}{1 - 2\rho\cos\theta + \rho^2} = \frac{2 - \rho(e^{i\theta} + e^{-i\theta})}{(1 - \rho e^{i\theta})(1 - \rho e^{-i\theta})}$$

$$= \frac{1}{1 - \rho e^{i\theta}} + \frac{1}{1 - \rho e^{-i\theta}}.$$

By the Binomial Theorem or actual division this becomes

$$1 + \rho e^{i\theta} + \rho^2 e^{2i\theta} + \rho^3 e^{3i\theta} + \dots +$$
$$+ 1 + \rho e^{-i\theta} + \rho^2 e^{-2i\theta} + \rho^3 e^{-3i\theta} + \dots$$
$$= 2 + \rho(e^{i\theta} + e^{-i\theta}) + \rho^2(e^{2i\theta} + e^{-2i\theta}) + \rho^3(e^{3i\theta} + e^{-3i\theta}) + \dots$$
$$= 2(1 + \rho\cos\theta + \rho^2\cos 2\theta + \rho^3\cos 3\theta + \dots).$$

Hence $f(\theta) = 1 + \rho\cos\theta + \rho^2\cos 2\theta + \rho^3\cos 3\theta + \dots$

2.—Expand $f(\theta) = \dfrac{\rho\cos\theta - \rho^2}{1 - 2\rho\cos\theta + \rho^2}$

We have

$$f(\theta) = \frac{1 - \rho\cos\theta}{1 - 2\rho\cos\theta + \rho^2} - 1.$$

Hence from Ex. 1 the expansion is

$$f(\theta) = \rho \cos \theta + \rho^2 \cos 2\theta + \rho^3 \cos 3\theta + \ldots$$

3.—Expand $f(\theta) = (1 - 2\rho \cos \theta + \rho^2)^{-1}$.

Since

$$\frac{1 - \rho \cos \theta}{1 - 2\rho \cos \theta + \rho^2} + \frac{\rho \cos \theta - \rho^2}{1 - 2\rho \cos \theta + \rho^2} = \frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2}$$

we have the expansion of $(1 - \rho^2)f(\theta)$ given by the sum of the series Ex. 1 and Ex. 2, that is,

$$(1 - \rho^2)f(\theta) = 1 + 2\rho \cos \theta + 2\rho^2 \cos 2\theta + 2\rho^3 \cos 3\theta + \ldots$$

$$\therefore f(\theta) = \frac{1}{1 - \rho^2}(1 + 2\rho \cos \theta + 2\rho^2 \cos 2\theta + 2\rho^3 \cos 3\theta + \ldots).$$

Note that by differentiation we obtain a new series:

$$f'(\theta) = -2\rho \sin \theta(1 - 2\rho \cos \theta + \rho^2)^{-2} = \frac{-1}{1 - \rho^2}(2\rho \sin \theta + 4\rho^2 \sin 2\theta + 6\rho^3 \sin 3\theta + \ldots).$$

4.—Expand $f(\theta) = (1 - 2\rho \cos \theta + \rho^2)^{\frac{1}{2}}$ We have

$$f(\theta) = (1 - \rho e^{i\theta})^{\frac{1}{2}}(1 - \rho e^{-i\theta})^{\frac{1}{2}}.$$

Expanding by the Binomial Theorem,

$$\begin{aligned}
f(\theta) &= (1 - \tfrac{1}{2}\rho e^{i\theta} - \tfrac{1}{8}\rho^2 e^{2i\theta} - \tfrac{1}{16}\rho^3 e^{3i\theta} - \tfrac{5}{128}\rho^4 e^{4i\theta} - \ldots)(1 - \tfrac{1}{2}\rho e^{-i\theta} \\
&\qquad - \tfrac{1}{8}\rho^2 e^{-2i\theta} - \tfrac{1}{16}\rho^3 e^{-3i\theta} - \tfrac{5}{128}\rho^4 e^{-4i\theta} - \ldots). \\
&= 1 - \tfrac{1}{2}\rho e^{i\theta} - \tfrac{1}{8}\rho^2 e^{2i\theta} - \tfrac{1}{16}\rho^3 e^{3i\theta} - \tfrac{5}{128}\rho^4 e^{4i\theta} - \ldots \\
&\quad - \tfrac{1}{2}\rho e^{-i\theta} + \tfrac{1}{4}\rho^2 + \tfrac{1}{16}\rho^3 e^{i\theta} + \tfrac{1}{32}\rho^4 e^{2i\theta} + \tfrac{5}{256}\rho^6 e^{3i\theta} + \ldots \\
&\quad - \tfrac{1}{8}\rho^2 e^{-2i\theta} + \tfrac{1}{16}\rho^3 e^{-i\theta} + \tfrac{1}{64}\rho^4 + \tfrac{1}{128}\rho^5 e^{i\theta} + \tfrac{5}{1024}\rho^6 e^{2i\theta} \\
&\qquad + \ldots \\
&\quad - \tfrac{1}{16}\rho^3 e^{-3i\theta} + \tfrac{1}{32}\rho^4 e^{-2i\theta} + \tfrac{1}{128}\rho^5 e^{-i\theta} + \tfrac{1}{256}\rho^6 \\
&\qquad + \tfrac{5}{2048}\rho^7 e^{i\theta} + \ldots \\
&\quad - \tfrac{5}{128}\rho^4 e^{-4i\theta} + \tfrac{5}{256}\rho^5 e^{-3i\theta} + \tfrac{5}{1024}\rho^6 e^{-2i\theta} \\
&\qquad + \tfrac{5}{2048}\rho^7 e^{-i\theta} + \tfrac{25}{16384}\rho^8 + \ldots \\
&= 1 + \tfrac{1}{4}\rho^2 + \tfrac{1}{64}\rho^4 + \tfrac{1}{256}\rho^6 + \ldots \\
&\quad + (-\tfrac{1}{2}\rho + \tfrac{1}{16}\rho^3 + \tfrac{1}{128}\rho^5 + \ldots)(e^{i\theta} + e^{-i\theta}) \\
&\quad + (-\tfrac{1}{8}\rho^2 + \tfrac{1}{32}\rho^4 + \tfrac{5}{1024}\rho^6 + \ldots)(e^{2i\theta} + e^{-2i\theta}) \\
&\quad + (-\tfrac{1}{16}\rho^3 + \tfrac{5}{256}\rho^5 + \ldots)(e^{3i\theta} + e^{-3i\theta}) \\
&\quad + (-\tfrac{5}{128}\rho^4 + \ldots)(e^{4i\theta} + e^{-4i\theta}) \\
&\quad + \ldots
\end{aligned}$$

Hence

$$f(\theta) = a_o + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta + \ldots$$

where
$$\begin{aligned} a_0 &= 1 + \tfrac{1}{4}\rho^2 + \tfrac{1}{64}\rho^4 + \dots \\ a_1 &= -\rho + \tfrac{1}{8}\rho^3 + \tfrac{1}{64}\rho^5 + \dots \\ a_2 &= -\tfrac{1}{4}\rho^2 + \tfrac{1}{16}\rho^4 + \tfrac{5}{512}\rho^6 + \dots \\ a_3 &= -\tfrac{1}{8}\rho^3 + \tfrac{5}{128}\rho^5 + \dots \\ a_4 &= -\tfrac{5}{64}\rho^4 + \dots \end{aligned}$$

Example.—Calculate a_0, a_1, a_2, a_3, a_4 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: 1·0279, − 0·3286, − ·0270, − ·0045, − ·0009.
$\rho = \frac{1}{4}$. Ans.: 1·0156, − 0·2481, − ·0154, − ·0019, − ·0002,
$\rho = \frac{1}{5}$. Ans.: 1·0100, − 0·1999, − ·0099, − ·0010, − ·0001.

Note that by differentiation we obtain a new series:

$$f'(\theta) = \rho \sin\theta(1 - 2\rho\cos\theta + \rho^2)^{-\frac{1}{2}} = -a_1 \sin\theta - 2a_2 \sin 2\theta - 3a_3 \sin 3\theta - \dots$$

Relation between the coefficients. We have, writing Δ for $1 - 2\rho\cos\theta + \rho^2$:

$$\Delta^{\frac{1}{2}} = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \dots$$

Multiply both sides by $\sin\theta$ and we get

$$\begin{aligned} \sin\theta\,.\,\Delta^{\frac{1}{2}} &= a_0 \sin\theta + a_1 \sin\theta\cos\theta + a_2 \sin\theta\cos 2\theta + a_3 \sin\theta\cos 3\theta + \dots \\ &= a_0 \sin\theta + \tfrac{1}{2}a_1 \sin 2\theta + \tfrac{1}{2}a_2(\sin 3\theta - \sin\theta) + \tfrac{1}{2}a_3(\sin 4\theta - \sin 2\theta) + \dots \\ &= (a_0 - \tfrac{1}{2}a_2)\sin\theta + (\tfrac{1}{2}a_1 - \tfrac{1}{2}a_3)\sin 2\theta + (\tfrac{1}{2}a_2 - \tfrac{1}{2}a_4)\sin 3\theta + \dots \\ &\quad + (\tfrac{1}{2}a_{n-1} - \tfrac{1}{2}a_{n+1})\sin n\theta + \dots \end{aligned}$$

The differentiated series is $f'(\theta) =$

$$\rho\sin\theta\,.\,\Delta^{-\frac{1}{2}} = -(a_1 \sin\theta + 2a_2 \sin 2\theta + 3a_3 \sin 3\theta + \dots).$$

Multiply both sides by $\dfrac{\Delta}{\rho} = \rho + \dfrac{1}{\rho} - 2\cos\theta = \lambda - 2\cos\theta$

where $\lambda = \rho + \dfrac{1}{\rho}$; we get:

$$\begin{aligned} \sin\theta.\,\Delta^{\frac{1}{2}} &= -a_1\lambda\sin\theta - 2a_2\lambda\sin 2\theta - 3a_3\lambda\sin 3\theta - \dots - na_n\lambda\sin n\theta - \dots \\ &\quad + 2a_1\sin\theta\cos\theta + 4a_2\sin 2\theta\cos\theta + 6a_3\sin 3\theta\cos\theta + \dots + 2na_n\sin n\theta\cos\theta + \dots \\ &= -\lambda a_1\sin\theta - 2\lambda a_2\sin 2\theta - 3\lambda a_3\sin 3\theta - \dots - n\lambda a_n\sin n\theta - \dots \\ &\quad + a_1\sin 2\theta + 2a_2(\sin 3\theta + \sin\theta) + 3a_3(\sin 4\theta + \sin 2\theta) + \dots \\ &\quad \dots + na_n\{\sin(n+1)\theta + \sin(n-1)\theta\} + \dots \\ &= (-\lambda a_1 + 2a_2)\sin\theta + (-2\lambda a_2 + a_1 + 3a_3)\sin 2\theta + (-3\lambda a_3 + 2a_2 + 4a_4)\sin 3\theta + \dots \\ &\quad \dots + \{-n\lambda a_n + (n-1)a_{n-1} + (n+1)a_{n+1}\}\sin n\theta + \dots \end{aligned}$$

This series for $\sin\theta\,.\,\Delta^{\frac{1}{2}}$ must be identical with the previous one, and it can be shown that the only way in which this identity can subsist is that the coefficients of corresponding sines coincide in both. Hence

$$
\begin{aligned}
a_o - \tfrac{1}{2}a_2 &= -\lambda a_1 + 2a_2 & \therefore\ a_2 &= \tfrac{2}{5}(a_o + \lambda a_1)\\
\tfrac{1}{2}a_1 - \tfrac{1}{2}a_3 &= -2\lambda a_2 + a_1 + 3a_3 & \therefore\ a_3 &= \tfrac{1}{7}(4\lambda a_2 - a_1)\\
\tfrac{1}{2}a_2 - \tfrac{1}{2}a_4 &= -3\lambda a_3 + 2a_2 + 4a_4 & \therefore\ a_4 &= \tfrac{1}{9}(6\lambda a_3 - 3a_2)\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\tfrac{1}{2}a_{n-1} - \tfrac{1}{2}a_{n+1} &= -n\lambda a_n + (n-1)a_{n-1} + (n+1)a_{n+1}
\end{aligned}
$$

$$
\therefore\ a_{n+1} = \frac{2n\lambda a_n - (2n-3)a_{n-1}}{2n+3}
$$

5.—Expand $f(\theta) = (1 - 2\rho\cos\theta + \rho^2)^{-\frac{1}{2}}$

We have $f(\theta) = (1 - \rho e^{i\theta})^{-\frac{1}{2}}(1 - \rho e^{-i\theta})^{-\frac{1}{2}}$.

Expanding by the Binomial Theorem, we get

$$
\begin{aligned}
f(\theta) = {} & (1 + \tfrac{1}{2}\rho e^{i\theta} + \tfrac{3}{8}\rho^2 e^{2i\theta} + \tfrac{5}{16}\rho^3 e^{3i\theta} + \tfrac{35}{128}\rho^4 e^{4i\theta} + \ .\ .\ .)\\
& \qquad (1 + \tfrac{1}{2}\rho e^{-i\theta} + \tfrac{3}{8}\rho^2 e^{-2i\theta} + \tfrac{5}{16}\rho^3 e^{-3i\theta} + \ .\ .\ .)\\
= {} & 1 + \tfrac{1}{2}\rho e^{i\theta} + \tfrac{3}{8}\rho^2 e^{2i\theta} + \tfrac{5}{16}\rho^3 e^{3i\theta} + \tfrac{35}{128}\rho^4 e^{4i\theta} + \ .\ .\ .\\
& + \tfrac{1}{2}\rho e^{-i\theta} + \tfrac{1}{4}\rho^2 + \tfrac{3}{16}\rho^3 e^{i\theta} + \tfrac{5}{32}\rho^4 e^{2i\theta} + \tfrac{35}{256}\rho^5 e^{3i\theta} + \ .\ .\ .\\
& + \tfrac{3}{8}\rho^2 e^{-2i\theta} + \tfrac{3}{16}\rho^3 e^{-i\theta} + \tfrac{9}{64}\rho^4 + \tfrac{15}{128}\rho^5 e^{i\theta} + \tfrac{105}{1024}\rho^6 e^{2i\theta} + \ .\ .\ .\\
& + \tfrac{5}{16}\rho^3 e^{-3i\theta} + \tfrac{5}{32}\rho^4 e^{-2i\theta} + \tfrac{15}{128}\rho^5 e^{-i\theta} + \tfrac{25}{256}\rho^6 + \ .\ .\ .\\
& + \tfrac{35}{128}\rho^4 e^{-4i\theta} + \tfrac{35}{256}\rho^5 e^{-3i\theta} + \tfrac{105}{1024}\rho^6 e^{-2i\theta} + \ .\ .\ .\\[1ex]
= {} & 1 + \tfrac{1}{4}\rho^2 + \tfrac{9}{64}\rho^4 + \tfrac{25}{256}\rho^6 + \ .\ .\ .\\
& + (\tfrac{1}{2}\rho + \tfrac{3}{16}\rho^3 + \tfrac{15}{128}\rho^5 + \ .\ .\ .)(e^{i\theta} + e^{-i\theta})\\
& + (\tfrac{3}{8}\rho^2 + \tfrac{5}{32}\rho^4 + \tfrac{105}{1024}\rho^6 + \ .\ .\ .)(e^{2i\theta} + e^{-2i\theta})\\
& + \tfrac{5}{16}\rho^3 + \tfrac{35}{256}\rho^5 + \ .\ .\ .)(e^{3i\theta} + e^{-3i\theta})\\
& + (\tfrac{35}{128}\rho^4 + \ .\ .\ .)(e^{4i\theta} + e^{-4i\theta}).
\end{aligned}
$$

Hence

$$
f(\theta) = b_o + b_1\cos\theta + b_2\cos 2\theta + b_3\cos 3\theta + b_4\cos 4\theta + \ .\ .\ .
$$

where

$$
\begin{aligned}
b_o &= 1 + \tfrac{1}{4}\rho^2 + \tfrac{9}{64}\rho^4 + \tfrac{25}{256}\rho^6 + \ .\ .\ .\\
b_1 &= \rho + \tfrac{3}{8}\rho^3 + \tfrac{15}{64}\rho^5\\
b_2 &= \tfrac{3}{4}\rho^2 + \tfrac{5}{16}\rho^4 + \tfrac{105}{512}\rho^6 + \ .\ .\ .\\
b_3 &= \tfrac{5}{8}\rho^3 + \tfrac{35}{128}\rho^5 + \ .\ .\ .\\
b_4 &= \tfrac{35}{64}\rho^4 + \ .\ .\ .
\end{aligned}
$$

Example.—Calculate b_o, b_1, b_2, b_3, b_4 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: 1·0294, ·3472, ·0874, ·0289, ·0066.
$\rho = \frac{1}{4}$. Ans.: 1·0161, ·2560, ·0478, ·0099, ·0022.
$\rho = \frac{1}{5}$. Ans.: 1·0102, ·2031, ·0305, ·0051, ·0009.

Note that by differentiation we get a new expansion:

$$f'(\theta) = -\rho \sin\theta(1 - 2\rho\cos\theta + \rho^2)^{-\frac{3}{2}} = -b_1 \sin\theta - 2b_2 \sin 2\theta \\ - 3b_3 \sin 3\theta - \ldots$$

Relation between the coefficients. We have:

$$\Delta^{-\frac{1}{2}} = b_o + b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta + b_4 \cos 4\theta + \ldots \\ + b_n \cos n\theta + \ldots$$

Multiply both sides by $\sin\theta$:

$$\begin{aligned} \sin\theta \,.\, \Delta^{-\frac{1}{2}} &= b_o \sin\theta + b_1 \sin\theta\cos\theta + b_2 \sin\theta\cos 2\theta \\ &\qquad + b_3 \sin\theta \cos 3\theta + \ldots \\ &= (b_o - \tfrac{1}{2}b_2)\sin\theta + (\tfrac{1}{2}b_1 - \tfrac{1}{2}b_3)\sin 2\theta + (\tfrac{1}{2}b_2 - \tfrac{1}{2}b_4)\sin 3\theta \\ &\qquad + \ldots + (\tfrac{1}{2}b_{n-1} - \tfrac{1}{2}b_{n+1})\sin n\theta + \ldots \end{aligned}$$

The differentiated series for $-f'(\theta) =$

$$\rho \sin\theta \,.\, \Delta^{-\frac{3}{2}} = b_1 \sin\theta + 2b_2 \sin 2\theta + 3b_3 \sin 3\theta + \ldots$$

Multiply both sides by $\dfrac{\Delta}{\rho} = \lambda - 2\cos\theta$ where as before $\lambda = \rho + \dfrac{1}{\rho}$, and we get

$$\begin{aligned} \sin\theta \,.\, \Delta^{-\frac{1}{2}} &= (\lambda b_1 - 2b_2)\sin\theta + (2\lambda b_2 - b_1 - 3b_3)\sin 2\theta \\ &\qquad + (3\lambda b_3 - 2b_2 - 4b_4)\sin 3\theta \\ &+ \ldots + \{n\lambda b_n - (n-1)b_{n-1} - (n+1)b_{n+1}\}\sin n\theta + \ldots \end{aligned}$$

Equating coefficients of corresponding sines in this and the previous series we get

$$\begin{aligned} b_o - \tfrac{1}{2}b_2 &= \lambda b_1 - 2b_2 & \therefore\ b_2 &= \tfrac{2}{3}(\lambda b_1 - b_o) \\ \tfrac{1}{2}b_1 - \tfrac{1}{2}b_3 &= 2\lambda b_2 - b_1 - 3b_3 & \therefore\ b_3 &= \tfrac{1}{5}(4\lambda b_2 - 3b_1) \\ \tfrac{1}{2}b_2 - \tfrac{1}{2}b_4 &= 3\lambda b_3 - 2b_2 - 4b_4 & \therefore\ b_4 &= \tfrac{1}{7}(6\lambda b_3 - 5b_2) \\ \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \end{aligned}$$

$$\tfrac{1}{2}b_{n-1} - \tfrac{1}{2}b_{n+1} = n\lambda b_n - (n-1)b_{n-1} - (n+1)b_{n+1}$$

$$\therefore\ b_{n+1} = \frac{2n\lambda b_n - (2n-1)b_{n-1}}{2n+1}.$$

6 —Expand $f(\theta) = (1 - 2\rho\cos\theta + \rho^2)^{-\frac{3}{2}}$.

We have $f(\theta) = (1 - \rho e^{i\theta})^{-\frac{3}{2}}(1 - \rho e^{-i\theta})^{-\frac{3}{2}}$, which on expansion by the Binomial Theorem becomes:

$$
\begin{aligned}
f(\theta) &= (1 + \tfrac{3}{2}\rho e^{i\theta} + \tfrac{15}{8}\rho^2 e^{2i\theta} + \tfrac{35}{16}\rho^3 e^{3i\theta} + \tfrac{315}{128}\rho^4 e^{4i\theta})(1 + \tfrac{3}{2}\rho e^{-i\theta} \\
&\qquad + \tfrac{15}{8}\rho^2 e^{-2i\theta} + \ldots). \\
&= 1 + \tfrac{3}{2}\rho e^{i\theta} + \tfrac{15}{8}\rho^2 e^{2i\theta} + \tfrac{35}{16}\rho^3 e^{3i\theta} + \tfrac{315}{128}\rho^4 e^{4i\theta} + \ldots \\
&+ \tfrac{3}{2}\rho e^{-i\theta} + \tfrac{9}{4}\rho^2 + \tfrac{45}{16}\rho^3 e^{i\theta} + \tfrac{105}{32}\rho^4 e^{2i\theta} + \tfrac{945}{256}\rho^5 e^{3i\theta} + \\
&+ \tfrac{15}{8}\rho^2 e^{-2i\theta} + \tfrac{45}{16}\rho^3 e^{-i\theta} + \tfrac{225}{64}\rho^4 + \tfrac{525}{128}\rho^5 e^{i\theta} + \tfrac{4725}{1024}\rho^6 e^{2i\theta} + \\
&+ \tfrac{35}{16}\rho^3 e^{-3i\theta} + \tfrac{105}{32}\rho^4 e^{-2i\theta} + \tfrac{525}{128}\rho^5 e^{-i\theta} + \tfrac{1225}{256}\rho^6 + \\
&+ \tfrac{315}{128}\rho^4 e^{-4i\theta} + \ldots \\
&= 1 + \tfrac{9}{4}\rho^2 + \tfrac{225}{64}\rho^4 + \tfrac{1225}{256}\rho^6 + \ldots \\
&+ (\tfrac{3}{2}\rho + \tfrac{45}{16}\rho^3 + \tfrac{525}{128}\rho^5 + \ldots)(e^{i\theta} + e^{-i\theta}) \\
&+ (\tfrac{15}{8}\rho^2 + \tfrac{105}{32}\rho^4 + \tfrac{4725}{1024}\rho^6 + \ldots)(e^{2i\theta} + e^{-2i\theta}) \\
&+ (\tfrac{35}{16}\rho^3 + \tfrac{945}{256}\rho^5 + \ldots)(e^{3i\theta} + e^{-3i\theta}) \\
&+ (\tfrac{315}{128}\rho^4 + \ldots)(e^{4i\theta} + e^{-4i\theta}).
\end{aligned}
$$

Hence

$$f(\theta) = c_o + c_1 \cos\theta + c_2 \cos 2\theta + c_3 \cos 3\theta + c_4 \cos 4\theta + \ldots$$

where

$$
\begin{aligned}
c_o &= 1 + \tfrac{9}{4}\rho^2 + \tfrac{225}{64}\rho^4 + \tfrac{1225}{256}\rho^6 + \ldots \\
c_1 &= 3\rho + \tfrac{45}{8}\rho^3 + \tfrac{525}{64}\rho^5 + \ldots \\
c_2 &= \tfrac{15}{4}\rho^2 + \tfrac{105}{16}\rho^4 + \ldots \\
c_3 &= \tfrac{35}{8}\rho^3 + \tfrac{945}{128}\rho^5 \\
c_4 &= \tfrac{315}{64}\rho^4 + \ldots
\end{aligned}
$$

Example.—Calculate c_o, c_1, c_2, c_3, c_4 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: 1·3000, 1·2422, ·4968, ·1928, ·0617
$\rho = \frac{1}{4}$. Ans.: 1·1553, ·8461, ·3359, ·0757, ·0195
$\rho = \frac{1}{5}$. Ans.: 1·0959, ·6436, ·1604, ·0586, ·0080

Note that by differentiation we get a new development:

$$
\begin{aligned}
f'(\theta) = -3\rho \sin\theta (1 - 2\rho\cos\theta + \rho^2)^{-\frac{5}{2}} &= -c_1 \sin\theta - 2c_2 \sin 2\theta \\
&\quad - 3c_3 \sin 3\theta - \ldots
\end{aligned}
$$

Relation between the coefficients. We have

$$
\begin{aligned}
\Delta^{-\frac{3}{2}} &= c_o + c_1 \cos\theta + c_2 \cos 2\theta + c_3 \cos 3\theta + \ldots \\
\therefore \sin\theta\,.\,\Delta^{-\frac{3}{2}} &= (c_o - \tfrac{1}{2}c_2)\sin\theta + (\tfrac{1}{2}c_1 - \tfrac{1}{2}c_3)\sin 2\theta \\
&\quad + (\tfrac{1}{2}c_2 - \tfrac{1}{2}c_4)\sin 3\theta + \ldots \\
&\quad + (\tfrac{1}{2}c_{n-1} - \tfrac{1}{2}c_{n+1})\sin n\theta + \ldots
\end{aligned}
$$

The differentiated series is

$$
\begin{aligned}
-f'(\theta) = 3\rho \sin\theta (1 - 2\rho\cos\theta + \rho^2)^{-\frac{5}{2}} &= c_1 \sin\theta + 2c_2 \sin 2\theta \\
&\quad + 3c_3 \sin 3\theta + \ldots
\end{aligned}
$$

Multiplying the left side by $\frac{\Delta}{\rho}$ and the right side by its equivalent $\lambda - 2\cos\theta$ where $\lambda = \rho + \frac{1}{\rho}$, we get

$$\begin{aligned}3\sin\theta\,.\,\Delta^{-\frac{3}{2}} &= (\lambda c_1 - 2c_2)\sin\theta + (2\lambda c_2 - c_1 - 3c_3)\sin 2\theta \\ &+ (3\lambda c_3 - 2c_2 - 4c_4)\sin 3\theta + \ldots \\ &+ \{n\lambda c_n - (n-1)c_{n-1} - (n+1)c_{n+1}\}\sin n\theta + \end{aligned}$$

Equating coefficients of corresponding sines in this and the previous series:

$$\begin{aligned}&3(c_0 - \tfrac{1}{2}c_2) = \lambda c_1 - 2c_2 && \therefore\ c_2 = 2\lambda c_1 - 6c_0 \\ &3(\tfrac{1}{2}c_1 - \tfrac{1}{2}c_3) = 2\lambda c_2 - c_1 - 3c_3 && \therefore\ c_3 = \tfrac{1}{3}(4\lambda c_2 - 5c_1) \\ &3(\tfrac{1}{2}c_2 - \tfrac{1}{2}c_4) = 3\lambda c_3 - 2c_2 - 4c_4 && \therefore\ c_4 = \tfrac{1}{5}(6\lambda c_3 - 7c_2) \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &3(\tfrac{1}{2}c_{n-1} - \tfrac{1}{2}c_{n+1}) = n\lambda c_n - (n-1)c_{n-1} - (n+1)c_{n+1} \\ &\qquad\qquad \therefore\ c_{n+1} = \frac{2n\lambda c_n - (2n+1)c_{n-1}}{2n-1}.\end{aligned}$$

7.—Expand $f(\theta) = (1 - 2\rho\cos\theta + \rho^2)^{-2}$

$$\begin{aligned}f(\theta) &= (1 - \rho e^{i\theta})^{-2}(1 - \rho e^{-i\theta})^{-2} \\ &= (1 + 2\rho e^{i\theta} + 3\rho^2 e^{2i\theta} + 4\rho^3 e^{3i\theta} + 5\rho^4 e^{4i\theta} + \ldots) \\ &\qquad (1 + 2\rho e^{-i\theta} + 3\rho^2 e^{-2i\theta} + \ldots) \\ &= 1 + 2\rho e^{i\theta} + 3\rho^2 e^{2i\theta} + 4\rho^3 e^{3i\theta} + 5\rho^4 e^{4i\theta} + \ldots \\ &+ 2\rho e^{-i\theta} + 4\rho^2 + 6\rho^3 e^{i\theta} + 8\rho^4 e^{2i\theta} + 10\rho^5 e^{3i\theta} + \ldots \\ &+ 3\rho^2 e^{-2i\theta} + 6\rho^3 e^{-i\theta} + 9\rho^4 + 12\rho^5 e^{i\theta} + 15\rho^6 e^{2i\theta} + \ldots \\ &+ 4\rho^3 e^{-3i\theta} + 8\rho^4 e^{-2i\theta} + 12\rho^5 e^{-i\theta} + 16\rho^6 + 20\rho^7 e^{i\theta} + \ldots \\ &+ 5\rho^4 e^{-4i\theta} + 10\rho^5 e^{-3i\theta} + 15\rho^6 e^{-2i\theta} + 20\rho^7 e^{-i\theta} + 25\rho^8 \\ &\qquad + \ldots \\ &= 1 + 4\rho^2 + 9\rho^4 + 16\rho^6 + 25\rho^8 + \ldots \\ &+ (2\rho + 6\rho^3 + 12\rho^5 + 20\rho^7 + \ldots)(e^{i\theta} + e^{-i}) \\ &+ (3\rho^2 + 8\rho^4 + 15\rho^6 + \ldots)(e^{2i\theta} + e^{-2i\theta}) \\ &+ (4\rho^3 + 10\rho^5 + 18\rho^7 + \ldots)(e^{3i\theta} + e^{-3i\theta}) \\ &+ (5\rho^4 + 12\rho^6 + \ldots)(e^{4i\theta} + e^{-4i\theta})\end{aligned}$$

Hence

$$f(\theta) = d_0 + d_1\cos\theta + d_2\cos 2\theta + d_3\cos 3\theta + d_4\cos 4\theta + \ldots$$

where

$$\begin{aligned}d_0 &= 1 + 4\rho^2 + 9\rho^4 + 16\rho^6 + 25\rho^8 + \ldots = \frac{1+\rho^2}{(1-\rho^2)^3} \\ d_1 &= 4\rho + 12\rho^3 + 24\rho^5 + 40\rho^7 + \ldots = 4\rho(1-\rho^2)^{-3} \\ d_2 &= 6\rho^2 + 16\rho^4 + 30\rho^6 + \ldots = 2(3\rho^2 - \rho^4)(1-\rho^2)^{-3} \\ d_3 &= 8\rho^3 + 20\rho^5 + 36\rho^7 + \ldots = 2(4\rho^3 - 2\rho^5)(1-\rho^2)^{-3} \\ d_4 &= 10\rho^4 + 24\rho^6 + \ldots = 2(5\rho^4 - 3\rho^6)(1-\rho^2)^{-3}\end{aligned}$$

Example.—Calculate d_o, d_1, d_2, d_3, d_4 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: 1·5820, 1·8984, ·9141, ·3951, ·1454
$\rho = \frac{1}{4}$. Ans.: 1·2883, 1·2112, ·4434, ·1522, ·0438
$\rho = \frac{1}{5}$. Ans.: 1·1754, ·9042, ·2675, ·0708, ·0175

Relation between the coefficients:

$$\Delta^{-2} = d_o + d_1 \cos\theta + d_2 \cos 2\theta + d_3 \cos 3\theta + d_4 \cos 4\theta + \ldots$$

Multiply both sides by $\sin\theta$:

$$\begin{aligned} \therefore \sin\theta \, . \, \Delta^{-2} = (d_o - \tfrac{1}{2}d_2) \sin\theta + (\tfrac{1}{2}d_1 - \tfrac{1}{2}d_3) \sin 2\theta \\ + (\tfrac{1}{2}d_2 - \tfrac{1}{2}d_4) \sin 3\theta + \ldots \\ + (\tfrac{1}{2}d_{n-1} - \tfrac{1}{2}d_{n+1}) \sin n\theta + \ldots \end{aligned}$$

By differentiation also $f'(\theta) =$

$$\begin{aligned} -2\Delta^{-3} \, . \, \frac{d\Delta}{d\theta} = -4\rho \sin\theta \, . \, \Delta^{-3} = -(d_1 \sin\theta + 2d_2 \sin 2\theta \\ + 3d_3 \sin 3\theta + \ldots) \end{aligned}$$

Multiply the left member by $\frac{\Delta}{\rho}$ and the right member by $\lambda - 2\cos\theta = \frac{\Delta}{\rho}$, and we get

$$\begin{aligned} 4 \sin\theta \, . \, \Delta^{-2} = (\lambda d_1 - 2d_2) \sin\theta + (2\lambda d_2 - d_1 - 3d_3) \sin 2\theta \\ + (3\lambda d_3 - 2d_2 - 4d_4) \sin 3\theta \\ + \ldots + \{n\lambda d_n - (n-1)d_{n-1} - (n+1)d_{n+1}\} \sin n\theta + \ldots \end{aligned}$$

Equating coefficients:

$$4(d_o - \tfrac{1}{2}d_2) = \lambda d_1 - 2d_2 \qquad \therefore d_1 = \frac{4d_o}{\lambda}$$

$$4(\tfrac{1}{2}d_1 - \tfrac{1}{2}d_3) = 2\lambda d_2 - d_1 - 3d_3 \qquad \therefore d_3 = 2\lambda d_2 - 3d_1$$

$$4(\tfrac{1}{2}d_2 - \tfrac{1}{2}d_4) = 3\lambda d_3 - 2d_2 - 4d_4 \qquad \therefore d_4 = \tfrac{1}{2}(3\lambda d_3 - 4d_2)$$

. . . .

$$4(\tfrac{1}{2}d_{n-1} - \tfrac{1}{2}d_{n+1}) = n\lambda d_n - (n-1)d_{n-1} - (n+1)d_{n+1}$$

$$\therefore d_{n+1} = \frac{n\lambda d_n - (n+1)d_{n-1}}{n-1}.$$

8.—If $\sin\phi = \rho \sin(\theta + \phi)$, obtain expansions for ϕ, $\dot{\phi}$, $\ddot{\phi}$ between the interval $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Written in exponential form the above equation becomes:

$$e^{i\phi} - e^{-i\phi} = \rho\{e^{i(\theta+\phi)} - e^{i(\theta-\phi)}\}.$$

Multiplying both sides of this equation by $e^{i\phi}$, we get

$$e^{2i\phi} - 1 = \rho\{e^{i(\theta + 2\phi)} - e^{-i\theta}\}$$
$$\therefore\ e^{2i\phi}(1 - \rho e^{i\theta}) = 1 - \rho e^{-i\theta}$$
$$\therefore\ e^{2i\phi} = (1 - \rho e^{-i\theta})/(1 - \rho e^{i\theta}).$$

Since ϕ lies between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, we can write

$$\begin{aligned}2i\phi &= \log(1 - \rho e^{-i\theta}) - \log(1 - \rho e^{i\theta})\\ &= -\rho e^{-i\theta} - \tfrac{1}{2}\rho^2 e^{-2i\theta} - \tfrac{1}{3}\rho^3 e^{-3i\theta} - \ .\ .\ .\\ &\quad + \rho e^{i\theta} + \tfrac{1}{2}\rho^2 e^{2i\theta} + \tfrac{1}{3}\rho^3 e^{3i\theta} + \ .\ .\ .\\ &= \rho\ .\ 2i\sin\theta + \tfrac{1}{2}\rho^2\ .\ 2i\sin 2\theta + \tfrac{1}{3}\rho^3\ .\ 2i\sin 3\theta + \ .\ .\ .\end{aligned}$$
$$\therefore\ \phi = \rho\sin\theta + \tfrac{1}{2}\rho^2\sin 2\theta + \tfrac{1}{3}\rho^3\sin 3\theta + \ .\ .\ .$$

By differentiation, assuming $\dot{\theta}$ constant, we get

$$\dot{\phi} = (\rho\cos\theta + \rho^2\cos 2\theta + \rho^3\cos 3\theta + \ .\ .\ .)\dot{\theta}$$
$$\ddot{\phi} = (-\rho\sin\theta - 2\rho^2\sin 2\theta - 3\rho^3\sin 3\theta - \ .\ .\ .)\dot{\theta}^2.$$

Note that the series for $\dot{\phi}/\dot{\theta}$ is identical with that of $f(\theta)$ in Example 2.

9.—Expand $f(\theta) = \dot{\phi}^2 = \dot{\theta}^2\left(\dfrac{\rho\cos\theta - \rho^2}{1 - 2\rho\cos\theta + \rho^2}\right)^2.$

We have $\dot{\phi} = \dot{\theta}\dfrac{\rho\cos\theta - \rho^2}{1 - 2\rho\cos\theta + \rho^2}$

$$= \tfrac{1}{2}\dot{\theta}\left(\frac{1 - \rho^2}{1 - 2\rho\cos\theta + \rho^2} - 1\right).$$

$$\begin{aligned}\therefore\ \dot{\phi}^2 &= \tfrac{1}{4}\dot{\theta}^2\{(1 - \rho^2)^2(1 - 2\rho\cos\theta + \rho^2)^{-2} - 2(1 - \rho^2)\\ &\qquad (1 - 2\rho\cos\theta + \rho^2)^{-1} + 1\}\\ &= \tfrac{1}{4}\dot{\theta}^2(1 - \rho^2)^2(d_0 + d_1\cos\theta + d_2\cos 2\theta + \ .\ .\ .)\\ &\quad - \tfrac{1}{2}\dot{\theta}^2(1 + 2\rho\cos\theta + 2\rho^2\cos 2\theta + 2\rho^3\cos 3\theta + \ .\ .\ .) + \tfrac{1}{4}\dot{\theta}^2.\end{aligned}$$

Hence

$$f(\theta) = (e_0 + e_1\cos\theta + e_2\cos 2\theta + e_3\cos 3\theta + e_4\cos 4\theta + \ .\ .\ .)\dot{\theta}^2,$$

where

$$\begin{aligned}e_0 &= \tfrac{1}{4}(1 - \rho^2)^2 d_0 - \tfrac{1}{4} = \tfrac{1}{2}(\rho^2 + \rho^4 + \rho^6 + \ .\ .\ .) = \tfrac{1}{2}\rho^2(1 - \rho^2)^{-1}\\ e_1 &= \tfrac{1}{4}(1 - \rho^2)^2 d_1 - \rho = \rho^3 + \rho^5 + \rho^7 + \ .\ .\ . = \rho^3(1 - \rho^2)^{-1}\\ e_2 &= \tfrac{1}{4}(1 - \rho^2)^2 d_2 - \rho^2 = \tfrac{1}{2}\rho^2 + \rho^4 + \rho^6 + \ .\ .\ . = \tfrac{1}{2}\rho^2(1 + \rho^2)(1 - \rho^2)^{-1}\\ e_3 &= \tfrac{1}{4}(1 - \rho^2)^2 d_3 - \rho^3 = \rho^3 + \rho^5 + \rho^7 + \ .\ .\ . = \rho^3(1 - \rho^2)^{-1}\\ e_4 &= \tfrac{1}{4}(1 - \rho^2)^2 d_4 - \rho^4 = \tfrac{3}{2}\rho^4 + \rho^6 + \rho^8 + \ .\ .\ . = \tfrac{1}{2}\rho^4(3 - \rho^2)(1 - \rho^2)^{-1}\end{aligned}$$

Example.—Calculate e_0, e_1, e_2, e_3, e_4 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: ·0624, ·0415, ·0693, ·0415, ·0200
$\rho = \frac{1}{4}$. Ans.: ·0333, ·0165, ·0353, ·0165, ·0061
$\rho = \frac{1}{5}$. Ans.: ·0208, ·0083, ·0217, ·0083, ·0025

Note that by differentiation we obtain if $\dot\theta$ be constant:

$$f'(\theta) = 2\dot\phi\ddot\phi = -\dot\theta^3(e_1 \sin\theta + 2e_2 \sin 2\theta + 3e_3 \sin 3\theta + \ldots).$$

The foregoing series may be developed with perhaps less labour by the following method. We have

$$\dot\phi = \dot\theta(\rho\cos\theta + \rho^2\cos 2\theta + \rho^3\cos 3\theta + \rho^4\cos 4\theta + \ldots)$$

$$\begin{aligned}\therefore \dot\phi^2 = \dot\theta^2\{&\rho^2\cos^2\theta + \rho^4\cos^2 2\theta + \rho^6\cos^2 3\theta + \rho^8\cos^2 4\theta + \ldots\\ &+ 2\rho^3\cos\theta\cos 2\theta + 2\rho^4\cos\theta\cos 3\theta + 2\rho^5\cos\theta\cos 4\theta + \ldots\\ &+ 2\rho^5\cos 2\theta\cos 3\theta + 2\rho^6\cos 2\theta\cos 4\theta + \ldots\\ &+ 2\rho^7\cos 3\theta\cos 4\theta + \ldots\}\end{aligned}$$

$$\begin{aligned}\therefore \dot\phi^2/\dot\theta^2 &= \tfrac{1}{2}\rho^2(\cos 2\theta + 1) + \tfrac{1}{2}\rho^4(\cos 4\theta + 1) + \tfrac{1}{2}\rho^6(\cos 6\theta + 1)\\ &+ \rho^3(\cos 3\theta + \cos\theta) + \rho^4(\cos 4\theta + \cos 2\theta) + \rho^5(\cos 5\theta + \cos 3\theta)\\ &+ \rho^5(\cos 5\theta + \cos\theta) + \rho^6(\cos 6\theta + \cos 2\theta) + \ldots\\ &+ \rho^7(\cos 7\theta + \cos\theta) + \ldots\\ &= \tfrac{1}{2}(\rho^2 + \rho^4 + \rho^6 + \ldots)\\ &+ (\rho^3 + \rho^5 + \rho^7 + \ldots)\cos\theta\\ &+ (\tfrac{1}{2}\rho^2 + \rho^4 + \rho^6 + \ldots)\cos 2\theta\\ &+ (\rho^3 + \rho^5 + \rho^7 + \ldots)\cos 3\theta\\ &+ (\tfrac{3}{2}\rho^4 + \rho^6 + \rho^8 + \ldots)\cos 4\theta + \ldots\end{aligned}$$

which is identical with the series obtained by another method.

10—If $\sin\phi = \rho\sin(\theta + \phi)$, show that

$$\sin\phi = \rho\sin\theta\,(1 - 2\rho\cos\theta + \rho^2)^{-\frac{1}{2}}.$$

Hence, from Example 4, write down the expansion of $\sin\phi$, $\phi\cos\phi$, and $\ddot\phi\cos\phi - \dot\phi^2\sin\phi$.

We have $\sin\phi = \rho(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= \rho\sin\theta(1 - \sin^2\phi)^{\frac{1}{2}} + \rho\cos\theta\sin\phi.$

$\therefore \sin\phi(1 - \rho\cos\theta) = \rho\sin\theta(1 - \sin^2\phi)^{\frac{1}{2}}$
$\therefore \sin^2\phi(1 - 2\rho\cos\theta + \rho^2\cos^2\theta) = \rho^2\sin^2\theta - \rho^2\sin^2\theta\sin^2\phi$
$\therefore \sin^2\phi(1 - 2\rho\cos\theta + \rho^2\cos^2\theta + \rho^2\sin^2\theta) = \rho^2\sin^2\theta.$

Hence $\sin\phi = \rho\sin\theta(1 - 2\rho\cos\theta + \rho^2)^{-\frac{1}{2}}$.

Referring to Example 4, we have

$$\sin\phi = -a_1\sin\theta - 2a_2\sin 2\theta - 3a_3\sin 3\theta - \ldots$$

Successive differentiation gives with θ constant

$$\phi \cos \phi = -\theta(a_1 \cos \theta + 4a_2 \cos 2\theta + 9a_3 \cos 3\theta + \ldots)$$
$$\ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi = \dot{\theta}^2(a_1 \sin \theta + 8a_2 \sin 2\theta + 27a_3 \sin 3\theta + \ldots)$$

11.—If $\sin \phi = \rho \sin(\theta + \phi)$, show that

$$\cos \phi = \tfrac{1}{2}(1 - 2\rho \cos \theta + \rho^2)^{\frac{1}{2}} + \tfrac{1}{2}(1 - \rho^2)(1 - 2\rho \cos \theta + \rho^2)^{-\frac{1}{2}}.$$

The given expression is

$$(1 - 2\rho \cos \theta + \rho^2)^{-\frac{1}{2}}\{\tfrac{1}{2}(1 - 2\rho \cos \theta + \rho^2) + \tfrac{1}{2}(1 - \rho^2)\}$$
$$= (1 - \rho \cos \theta)(1 - 2\rho \cos \theta + \rho^2)^{-\frac{1}{2}}.$$

Now we have

$$\cos^2 \phi = 1 - \sin^2 \phi = 1 - \frac{\rho^2 \sin^2 \theta}{1 - 2\rho \cos \theta + \rho^2}. \qquad \text{(Ex. 10)}$$
$$= \frac{1 - 2\rho \cos \theta + \rho^2 \cos^2 \theta}{1 - 2\rho \cos \theta + \rho^2}$$
$$\therefore \cos \phi = \frac{1 - \rho \cos \theta}{(1 - 2\rho \cos \theta + \rho^2)^{\frac{1}{2}}}$$
$$= \tfrac{1}{2}(1 - 2\rho \cos \theta + \rho^2)^{\frac{1}{2}} + \tfrac{1}{2}(1 - \rho^2)(1 - 2\rho \cos \theta + \rho^2)^{-\frac{1}{2}}$$

From Examples, 4, 5, we can now write down the expansion of $\cos \phi$:

$$\cos \phi = \tfrac{1}{2}(a_o + a_1 \cos \theta + a_2 \cos 2\theta + \ldots) + \tfrac{1}{2}(1 - \rho^2)(b_o + b_1 \cos \theta + b_2 \cos 2\theta + \ldots)$$
$$\therefore \cos \phi = f_o + f_1 \cos \theta + f_2 \cos 2\theta + f_3 \cos 3\theta + f_4 \cos 4\theta + \ldots$$

where

$$f_o = \tfrac{1}{2}(a_o + b_o - \rho^2 b_o)$$
$$f_1 = \tfrac{1}{2}(a_1 + b_1 - \rho^2 b_1)$$
$$f_2 = \tfrac{1}{2}(a_2 + b_2 - \rho^2 b_2)$$
$$f_3 = \tfrac{1}{2}(a_3 + b_3 - \rho^2 b_3)$$
$$f_4 = \tfrac{1}{2}(a_4 + b_4 - \rho^2 b_4).$$

Example.—Calculate f_o, f_1, f_2, f_3, f_4 for the following values of ρ:—

$\rho = \frac{1}{3}$. Ans.: ·9714, − ·0100, ·0253, ·0106, ·0028
$\rho = \frac{1}{4}$. Ans.: ·9841, − ·0040, ·0147, ·0037, ·0011
$\rho = \frac{1}{5}$. Ans.: ·9899, − ·0024, ·0097, ·0025, ·0004.

Note that by differentiation we obtain the new series

$\phi \sin \phi = \dot{\theta}(f_1 \sin \theta + 2f_2 \sin 2\theta + 3f_3 \sin 3\theta + \ldots)$ and if $\dot{\theta}$ be constant

$$\ddot{\phi} \sin \phi + \phi^2 \cos \phi = \theta^2(f_1 \cos \theta + 4f_2 \cos 2\theta + 9f_3 \cos 3\theta + \ldots)$$

12.—Expand $f(\theta) = \dot{\phi}^2 \sin\phi$ where $\dot{\phi}^2$ and $\sin\phi$ are defined as in Examples 9 and 10.

From Examples 9 and 10 we have, writing ω for $\dot{\theta}$,

$$\frac{1}{\omega^2}4f(\theta) = \frac{(1-\rho^2)^2\rho\sin\theta}{(1-2\rho\cos\theta+\rho^2)^{\frac{5}{2}}} + \frac{\rho\sin\theta}{(1-2\rho\cos\theta+\rho^2)^{\frac{1}{2}}} - \frac{2(1-\rho^2)\rho\sin\theta}{(1-2\rho\cos\theta+\rho^2)^{\frac{3}{2}}}$$

$$\begin{aligned}&= \tfrac{1}{3}(1-\rho^2)^2(c_1\sin\theta + 2c_2\sin 2\theta + \ldots) \quad . \quad \text{(Ex. 6)}\\ &\quad - a_1\sin\theta - 2a_2\sin 2\theta - \ldots \quad . \quad . \quad \text{(Ex, 4)}\\ &\quad - 2(1-\rho^2)(b_1\sin\theta + 2b_2\sin 2\theta + \ldots) \quad \text{(Ex. 5)}\end{aligned}$$

$$\therefore f(\theta) = \omega^2(g_1\sin\theta + g_2\sin 2\theta + g_3\sin 3\theta + g_4\sin 4\theta + \ldots)$$

where

$$\begin{aligned}g_1 &= \tfrac{1}{12}(1-\rho^2)^2c_1 - \tfrac{1}{4}a_1 - \tfrac{1}{2}(1-\rho^2)b_1\\ g_2 &= \tfrac{2}{12}(1-\rho^2)^2c_2 - \tfrac{1}{4}\cdot 2a_2 - \tfrac{1}{2}(1-\rho^2)2b_2\\ g_3 &= \tfrac{3}{12}(1-\rho^2)^2c_3 - \tfrac{1}{4}\cdot 3a_3 - \tfrac{1}{2}(1-\rho^2)3b_3\\ g_4 &= \tfrac{4}{12}(1-\rho^2)^2c_4 - \tfrac{1}{4}\cdot 4a_4 - \tfrac{1}{2}(1-\rho^2)4b_4.\end{aligned}$$

Example.—Calculate g_1, g_2, g_3, g_4 for the following values of ρ :—

$\rho = \frac{1}{3}$. Ans. :
$\rho = \frac{1}{4}$. Ans. :
$\rho = \frac{1}{5}$. Ans. :

13.—If $b = c(1 - 2\rho\cos\theta + \rho^2)^{\frac{1}{2}}$ and $\ddot{\phi} = -\omega^2\rho(1-\rho^2)\sin\theta(1-2\rho\cos\theta+\rho^2)^{-2}$, write down the expansion of $f(\theta) = b\dot{\phi}$:

$$\begin{aligned}b\dot{\phi} &= -\omega^2c\rho(1-\rho^2)\sin\theta(1-2\rho\cos\theta+\rho^2)^{-\frac{3}{2}}\\ &= -\omega^2c(1-\rho^2)(b_1\sin\theta + 2b_2\sin 2\theta + 3b_3\sin 3\theta + \ldots)\\ &\hfill \text{(Ex. 5)}\end{aligned}$$

14.—Write down the expansion of $f(\theta) = b\ddot{\phi}\dot{\phi}$ where b, $\ddot{\phi}$, and $\dot{\phi}$ are defined as in Examples 13 and 9.

$$\begin{aligned}\frac{1}{\omega^3}b\ddot{\phi}\dot{\phi} &= -c\rho\sin\theta(1-\rho^2)(1-2\rho\cos\theta+\rho^2)^{-\frac{3}{2}}\{\tfrac{1}{2}(1-\rho^2)\\ &\qquad (1-2\rho\cos\theta+\rho^2)^{-1} - 1\}\\ &= -\tfrac{1}{2}c\rho\sin\theta(1-\rho^2)^2(1-2\rho\cos\theta+\rho^2)^{-\frac{5}{2}} + \tfrac{1}{2}c\rho\sin\theta\\ &\qquad (1-\rho^2)(1-2\rho\cos\theta+\rho^2)^{-\frac{3}{2}}\\ &= -\tfrac{1}{6}c(1-\rho^2)^2(c_1\sin\theta + 2c_2\sin 2\theta + 3c_3\sin 3\theta + \ldots) \text{ Ex. 6}\\ &\quad + \tfrac{1}{2}c(1-\rho^2)(b_1\sin\theta + 2b_2\sin 2\theta + 3b_3\sin 3\theta + \ldots) \text{ Ex. 5}\end{aligned}$$

$$\therefore f(\theta) = \omega^3(r_1\sin\theta + r_2\sin 2\theta + r_3\sin 3\theta + \ldots).$$

15.—Expand $f(\theta) = \frac{1}{\rho}(1 - \rho^2\sin^2\theta)^{\frac{1}{2}}$.

We have

$$(1 + h)^{-2}(1 + he^{2i\theta})(1 + he^{-2i\theta}) = 1 - \frac{4h}{(1 + h)^2}\sin^2\theta.$$

Hence

$$1 - \rho^2 \sin^2\theta = (1 + h)^{-2}(1 + he^{2i\theta})(1 + he^{-2i\theta})$$

where $\rho^2 = 4h/(1 + h)^2$. Note that putting $\rho = \sin\alpha$ gives $h = \tan^2\frac{\alpha}{2}$. This gives a convenient method for calculating h for given values of ρ. Otherwise, solving for h we get the equation :—

$$h^2\rho^2 - h(4 - 2\rho^2) + \rho^2 = 0$$

$$\therefore h = \frac{1}{2\rho^2}(4 - 2\rho^2 \pm 4\sqrt{1 - \rho^2})$$

$$= \tfrac{1}{4}\rho^2 + \tfrac{1}{8}\rho^4 + \tfrac{5}{64}\rho^6 + \ldots$$

Example.—Calculate h for the following values of ρ :—

Ans. : $\rho = \frac{1}{3}$, $h = \cdot 0294$, $\rho = \frac{1}{4}$, $h = \cdot 0161$, $\rho = \frac{1}{5}$, $h = \cdot 0102$.

Returning to the expansion, we have

$$\rho f(\theta) = (1 + h)^{-1}(1 + he^{2i\theta})^{\frac{1}{2}}(1 + he^{-2i\theta})^{\frac{1}{2}}$$

$$(1 + he^{2i\theta})^{\frac{1}{2}}(1 + he^{-2i\theta})^{\frac{1}{2}} = p_0' + p_2'\cos 2\theta + p_4'\cos 4\theta + p_6'\cos 6\theta + \ldots$$

where

$$\begin{aligned}
p_0' &= 1 + \tfrac{1}{4}h^2 + \tfrac{1}{64}h^4 + \ldots \\
p_2' &= h - \tfrac{1}{8}h^3 - \tfrac{1}{64}h^5 \\
p_4' &= -\tfrac{1}{4}h^2 + \tfrac{1}{16}h^4 + \tfrac{5}{512}\rho^6 \\
p_6' &= \tfrac{1}{8}h^3 - \tfrac{5}{128}h^5 - \\
p_8' &= -\tfrac{5}{64}h^4 + \ldots
\end{aligned}$$

This result is obtained from Example 4 by writing 2θ for θ and $-h$ for ρ in the expansion there found. We have, therefore,

$$f(\theta) = \frac{1}{\rho(1 + h)}(p_0' + p_2'\cos 2\theta + p_4'\cos 4\theta + \ldots)$$

$$= q_0 + q_2\cos 2\theta + q_4\cos 4\theta + \ldots$$

where $q_n = p_n'/\rho(1 + h)$.

Example.—Calculate q_0, q_2, q_4, q_6 for the following values of ρ :—

$\rho = \frac{1}{3}$. Ans. : $2{\cdot}9150$, $\cdot 0857$, $-\cdot 0006$, $\cdot 000008$

$\rho = \frac{1}{4}$. Ans. : $3{\cdot}9367$, $\cdot 0635$, $-\cdot 00025$, $\cdot 000002$

$\rho = \frac{1}{5}$. Ans. : $4{\cdot}9496$, $\cdot 0505$, $-\cdot 000125$, $\cdot 0000006$.

16.—Expansion of $f(\theta) = \frac{1}{\rho}(1 - \rho^2 \sin^2 \theta)^{\frac{1}{2}}$ by another method.

Expanding at once by the Binomial Theorem, we get

$$\rho f(\theta) = 1 - \tfrac{1}{2}\rho^2 \sin^2 \theta - \tfrac{1}{8}\rho^4 \sin^4 \theta - \tfrac{1}{16}\rho^6 \sin^6 \theta - \ldots$$

$$= 1 - \tfrac{1}{2}\rho^2\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 - \tfrac{1}{8}\rho^4\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^4 - \tfrac{1}{16}\rho^6\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^6 - \ldots$$

$$\begin{aligned} = 1 &- \tfrac{1}{2}\rho^2(-\tfrac{1}{4})(e^{2i\theta} + e^{-2i\theta} - 2) \\ &- \tfrac{1}{8}\rho^4(\tfrac{1}{16})\{e^{4i\theta} + e^{-4i\theta} - 4(e^{2i\theta} + e^{-2i\theta}) + 6\} \\ &- \tfrac{1}{16}\rho^6(-\tfrac{1}{64})\{e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} \\ &\qquad + e^{-2i\theta}) - 20\} \\ &- \tfrac{5}{128}\rho^8(\tfrac{1}{256})\{e^{8i\theta} + e^{-8i\theta} - 8(e^{6i\theta} + e^{-6i\theta}) + \ldots\} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

$$\begin{aligned} = 1 &- \tfrac{1}{4}\rho^2 - \tfrac{3}{64}\rho^4 - \tfrac{5}{256}\rho^6 - \ldots \\ &+ (\tfrac{1}{8}\rho^2 + \tfrac{1}{32}\rho^4 + \tfrac{15}{1024}\rho^6 + \ldots)(e^{2i\theta} + e^{-2i\theta}) \\ &+ (-\tfrac{1}{128}\rho^4 - \tfrac{3}{512}\rho^6 - \ldots)(e^{4i\theta} + e^{-4i\theta}) \\ &+ (\tfrac{1}{1024}\rho^6 + \ )(e^{6i\theta} + e^{-6i\theta}). \end{aligned}$$

$$\therefore f(\theta) = q_0 + q_2 \cos 2\theta + q_4 \cos 4\theta + q_6 \cos 6\theta + \ldots$$

where

$$\begin{aligned} q_0 &= \frac{1}{\rho} - \tfrac{1}{4}\rho - \tfrac{3}{64}\rho^3 - \tfrac{5}{256}\rho^5 - \ldots \\ q_2 &= \tfrac{1}{4}\rho + \tfrac{1}{16}\rho^3 + \tfrac{15}{512}\rho^5 + \ldots \\ q_4 &= -\tfrac{1}{64}\rho^3 - \tfrac{3}{256}\rho^5 - \ldots \\ q_6 &= \tfrac{1}{512}\rho^5 + \ldots \\ q_8 &= -\tfrac{5}{16384}\rho^7 + \ldots \end{aligned}$$

Exercise.—If $p_n = n^2 q_n$, show that $p_2 + p_4 + p_6 + \ldots = \rho$

We have $f''(0) = -4q_2 - 16q_4 - 36q_6 - \ldots$

We have also

$$f''(\theta) = -\rho \cos 2\theta(1 - \rho^2 \sin^2 \theta)^{-\frac{1}{2}} - \rho^3 \sin^2 \theta \cos^2 \theta(1 - \rho^2 \sin^2 \theta)^{-\frac{3}{2}}$$

from which $f''(0) = -\rho$, whence result follows. See Art. 17 for meaning of p_n.

Show

$$q_0 + q_2 + q_4 + \ldots = \frac{1}{\rho}.$$

Show

$$f''(0^\circ) + f''(120^\circ) + f''(240^\circ) = 3(p_6 + p_{12} + p_{18} + \ldots)$$
$$= -\rho - \rho(1 - \tfrac{3}{4}\rho^2)^{-\frac{1}{2}} + \tfrac{3}{8}\rho^3(1 - \tfrac{3}{4}\rho^2)^{-\frac{3}{2}}.$$

Relation between the coefficients. We have

$$f(\theta) = (1 - \rho^2 \sin^2\theta)^{\frac{1}{2}}/\rho = q_0 + q_2 \cos 2\theta + q_4 \cos 4\theta + q_6 \cos 6\theta + \ldots + q_n \cos n\theta + \ldots$$

Multiplying both sides by sin 2θ:

$$\sin 2\theta(1 - \rho^2 \sin^2\theta)^{\frac{1}{2}}/\rho = q_0 \sin 2\theta + q_2 \sin 2\theta \cos 2\theta + q_4 \sin 2\theta \cos 4\theta + q_6 \sin 2\theta \cos 6\theta + \ldots$$
$$= q_0 \sin 2\theta + \tfrac{1}{2}q_2 \sin 4\theta + \tfrac{1}{2}q_4(\sin 6\theta - \sin 2\theta) + \tfrac{1}{2}q_6(\sin 8\theta - \sin 4\theta) + \ldots + \tfrac{1}{2}q_n\{\sin(n+2)\theta - \sin(n-2)\theta\} + \ldots$$
$$= (q_0 - \tfrac{1}{2}q_4)\sin 2\theta + (\tfrac{1}{2}q_2 - \tfrac{1}{2}q_6)\sin 4\theta + (\tfrac{1}{2}q_4 - \tfrac{1}{2}q_8)\sin 6\theta + \ldots + (\tfrac{1}{2}q_{n-2} - \tfrac{1}{2}q_{n+2})\sin n\theta + \ldots$$

Differentiating $f(\theta)$, we get

$$f'(\theta) = \frac{-\rho \sin 2\theta}{2(1 - \rho^2 \sin^2\theta)^{\frac{1}{2}}} = -(2q_2 \sin 2\theta + 4q_4 \sin 4\theta + \ldots + nq_n \sin n\theta + \ldots).$$

Multiplying both sides of this equation by $-2\rho^{-2}(1-\rho^2\sin^2\theta)$ and remembering that $2\rho^{-2}(1 - \rho^2\sin^2\theta) = 2\rho^{-2} - 1 + \cos 2\theta = \lambda + \cos 2\theta$ where $\lambda = 2\rho^{-2} - 1$ $\left(\text{note also } \lambda = \tfrac{1}{2}\left(h + \dfrac{1}{h}\right)\right)$ we get

$$\rho^{-1}\sin 2\theta(1-\rho^2\sin^2\theta)^{\frac{1}{2}} = (\lambda+\cos 2\theta)(2q_2 \sin 2\theta + 4q_4 \sin 4\theta + \ldots + nq_n \sin n\theta + \ldots)$$
$$= 2\lambda q_2 \sin 2\theta + 4\lambda q_4 \sin 4\theta + 6\lambda q_6 \sin 6\theta + \ldots + n\lambda q_n \sin n\theta + \ldots$$
$$+ 2q_2 \sin 2\theta \cos 2\theta + 4q_4 \sin 4\theta \cos 2\theta + 6q_6 \sin 6\theta \cos 2\theta + \ldots + nq_n \sin n\theta \cos 2\theta + \ldots$$
$$= 2\lambda q_2 \sin 2\theta + 4\lambda q_4 \sin 4\theta + 6\lambda q_6 \sin 6\theta + n\lambda q_n \sin n\theta + \ldots$$
$$+ q_2 \sin 4\theta + 2q_4(\sin 6\theta + \sin 2\theta) + 3q_6(\sin 8\theta + \sin 4\theta) + \ldots + \tfrac{1}{2}nq_n\{\sin(n+2)\theta + \sin(n-2)\theta\}$$
$$= (2\lambda q_2 + 2q_4)\sin 2\theta + (4\lambda q_4 + q_2 + 3q_6)\sin 4\theta + (6\lambda q_6 + 2q_4 + 4q_8)\sin 6\theta + \ldots$$
$$+ \left(n\lambda q_n + \frac{n-2}{2}q_{n-2} + \frac{n+2}{2}q_{n+2}\right)\sin n\theta + \ldots$$

Equating corresponding sine coefficients in this and the previous series, we get

$$\begin{aligned}
&q_o - \tfrac{1}{2}q_4 = 2\lambda q_2 + 2q_4 && \therefore q_4 = \tfrac{1}{5}(2q_o - 4\lambda q_2)\\
&\tfrac{1}{2}q_2 - \tfrac{1}{2}q_6 = 4\lambda q_4 + q_2 + 3q_6 && \therefore q_6 = \tfrac{1}{7}(-q_2 - 8\lambda q_4)\\
&\tfrac{1}{2}q_4 - \tfrac{1}{2}q_8 = 6\lambda q_6 + 2q_4 + 4q_8 && \therefore q_8 = \tfrac{1}{9}(-3q_4 - 12\lambda q_6)
\end{aligned}$$

$$\tfrac{1}{2}q_{n-2} - \tfrac{1}{2}q_{n+2} = n\lambda q_n + \frac{n-2}{2}q_{n-2} + \frac{n+2}{2}q_{n+2}$$

$$\therefore q_{n+2} = \frac{-2n\lambda q_n - (n-3)q_{n-2}}{n+3}$$

Note that multiplying all coefficients by a constant factor will not alter the above relation which therefore exists between the displacement coefficients for piston (section 38). Further, the acceleration coefficients p_n are related to the displacement coefficients q_n through the equation

$$p_n = n^2 q_n \quad \text{or} \quad q_n = p_n/n^2. \qquad \text{(Sect. 17)}$$

Therefore, substituting in the general formula above, we get the relation between the acceleration coefficients to be given by

$$\frac{p_{n+2}}{(n+2)^2} = \frac{-\dfrac{2n\lambda p_n}{n^2} - \dfrac{(n-3)p_{n-2}}{(n-2)^2}}{n+3}$$

or

$$\begin{aligned}
p_{n+2} &= -\frac{(n+2)^2}{n+3}\left\{\frac{2\lambda}{n}p_n + \frac{n-3}{(n-2)^2}p_{n-2}\right\}\\
\therefore p_4 &= \tfrac{16}{5}(-\lambda p_2 + q_o)\\
p_6 &= \tfrac{36}{7}(-\tfrac{1}{2}\lambda p_4 - \tfrac{1}{4}p_2)\\
p_8 &= \tfrac{64}{9}(-\tfrac{1}{3}\lambda p_6 - \tfrac{3}{16}p_4).
\end{aligned}$$

17.—Expand $f(\theta) = \dot{\phi} = \rho\omega\cos\theta(1 - \rho^2\sin^2\theta)^{-\frac{1}{2}}$.

From Example 15 we have $1 - \rho^2 \sin^2\theta = (1 + h)^{-2}(1 + 2h\cos 2\theta + h^2)$ where $\rho^2 = 4h/(1 + h)^2$. Hence

$$\begin{aligned}
f(\theta) &= \rho\omega\cos\theta(1 + h)(1 + 2h\cos 2\theta + h^2)^{-\frac{1}{2}}\\
&= \rho\omega\cos\theta(1 + h)(t_o + t_2\cos 2\theta + t_4\cos 4\theta + t_6\cos 6\theta + \ldots)
\end{aligned}$$

where

$$\begin{aligned}
t_o &= 1 + \tfrac{1}{4}h^2 + \tfrac{9}{64}h^4 + \ldots\\
t_2 &= -h - \tfrac{3}{8}h^3 - \tfrac{15}{64}h^5 - \ldots\\
t_4 &= \tfrac{3}{4}h^2 + \tfrac{5}{16}h^4 + \tfrac{105}{512}h^6 + \ldots\\
t_6 &= -\tfrac{5}{8}h^3 - \tfrac{35}{128}h^5 - \ldots\\
t_8 &= \tfrac{35}{64}h^4 +
\end{aligned}$$

(This can be written down at once by setting $2\theta = \theta$ and $-h = \rho$ in Example 5.)

$$
\begin{aligned}
\therefore f(\theta) &= \rho\omega(1 + h)(t_o \cos\theta + t_2 \cos\theta \cos 2\theta + t_4 \cos\theta \cos 4\theta \\
&\qquad + \ldots) \\
&= \rho\omega(1 + h)\{t_o \cos\theta + \tfrac{1}{2}t_2(\cos 3\theta + \cos\theta) + \tfrac{1}{2}t_4(\cos 5\theta \\
&\qquad + \cos 3\theta) + \tfrac{1}{2}t_6(\cos 7\theta + \cos 5\theta) + \ldots\} \\
&= \omega\{\rho(t_o + \tfrac{1}{2}t_2)(1 + h)\cos\theta + \rho(\tfrac{1}{2}t_2 + \tfrac{1}{2}t_4)(1 + h)\cos 3\theta \\
&\qquad + \ldots\} \\
&= \omega(u_1 \cos\theta + u_3 \cos 3\theta + u_5 \cos 5\theta + \ldots).
\end{aligned}
$$

Note that by differentiation we get a new series:

$$f'(\theta) = \ddot{\phi} = -\omega^2(u_1 \sin\theta + 3u_3 \sin 3\theta + 5u_5 \sin 5\theta + \ldots)$$

Otherwise, expanding by the Binomial Theorem, we get

$$\dot{\phi} = \rho\omega\cos\theta(1 + \tfrac{1}{2}\rho^2 \sin^2\theta + \tfrac{3}{8}\rho^4 \sin^4\theta + \tfrac{5}{16}\rho^6 \sin^6\theta + \ldots)$$

$$
\begin{aligned}
&= \rho\omega\Big\{\frac{e^{i\theta} + e^{-i\theta}}{2} + \tfrac{1}{2}\rho^2 \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{2i\theta} + e^{-2i\theta} - 2}{-4} \\
&\qquad + \tfrac{3}{8}\rho^4 \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{4i\theta} + e^{-4i\theta} - 4(e^{2i\theta} + e^{-2i\theta}) + 6}{16} \\
&+ \tfrac{15}{16}\rho^6 \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) - 20}{-64} \ldots + \Big\} \\
&= \rho\omega\{(e^{i\theta} + e^{-i\theta})(\tfrac{1}{2} + \tfrac{1}{16}\rho^2 + \tfrac{3}{128}\rho^4 + \tfrac{25}{2048}\rho^6 + \ldots) \\
&\qquad + (e^{3i\theta} + e^{-3i\theta})(\tfrac{1}{16}\rho^2 - \tfrac{9}{256}\rho^4 - \tfrac{45}{2048}\rho^6 - \ldots) \\
&\qquad + (e^{5i\theta} + e^{-5i\theta})(\tfrac{3}{256}\rho^4 + \tfrac{25}{2048}\rho^6 + \ldots) \\
&\qquad + (e^{7i\theta} + e^{-7i\theta})(-\tfrac{5}{2048}\rho^6 - \ldots)\} \\
&= \omega\,(u_1 \cos\theta + u_3 \cos 3\theta + u_5 \cos 5\theta + \ldots)
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 &= \rho + \tfrac{1}{8}\rho^3 + \tfrac{3}{64}\rho^5 + \ldots \\
u_3 &= -\tfrac{1}{8}\rho^3 - \tfrac{9}{128}\rho^5 - \ldots \\
u_5 &= \tfrac{3}{128}\rho^5 + \tfrac{25}{1024}\rho^7 + \ldots
\end{aligned}
$$

Example.—Calculate u_1, u_3, u_5 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: ·3379, — ·0046, ·0001
$\rho = \frac{1}{4}$. Ans.: ·2520, — ·0020, —
$\rho = \frac{1}{5}$. Ans.: ·2010, — ·0010, —

18.—Expand $f(\theta) = \dot{\phi}^2 = \rho^2\omega^2 \cos^2\theta(1 - \rho^2 \sin^2\theta)^{-1}$.

As in Example 17, we have

$$f(\theta) = \rho^2\omega^2 \cos^2\theta(1 + h)^2(1 + 2h\cos 2\theta + h^2)^{-1}$$
$$= 2\omega^2(h + h\cos 2\theta)(1 + 2h\cos 2\theta + h^2)^{-1}$$
$$= 2\omega^2\left\{\frac{1 + h\cos 2\theta}{1 + 2h\cos 2\theta + h^2} - \frac{1 - h}{1 + 2h\cos 2\theta + h^2}\right\}$$
$$= 2\omega^2(1 - h\cos 2\theta + h^2\cos 4\theta - h^3\cos 6\theta + \ldots)$$
$$- 2\omega^2(1 + h)^{-1}(1 - 2h\cos 2\theta + 2h^2\cos 4\theta - 2h^3\cos 6\theta + \ldots)$$
$$= \omega^2(s_o + s_2\cos 2\theta + s_4\cos 4\theta + s_6\cos 6\theta + \ldots).$$

By differentiation

$$2\dot{\phi}\ddot{\phi} = -\omega^3(2s_2\sin 2\theta + 4s_4\sin 4\theta + 6s_6\sin 6\theta + \ldots).$$

Otherwise

$$\dot{\phi}^2 = \rho^2\omega^2\cos^2\theta(1 - \rho^2\sin^2\theta)^{-1}$$
$$= \rho^2\omega^2\cos^2\theta(1 + \rho^2\sin^2\theta + \rho^4\sin^4\theta + \rho^6\sin^6\theta + \ldots)$$
$$= \frac{1}{4}\rho^2\omega^2(e^{2i\theta} + e^{-2i\theta} + 2)\{1 + \frac{\rho^2}{-4}(e^{2i\theta} + e^{-2i\theta} - 2)$$
$$+ \frac{\rho^4}{16}(e^{4i\theta} + e^{-4i\theta} - 4e^{2i\theta} - 4e^{-2i\theta} + 6)$$
$$+ \frac{\rho^6}{-64}(e^{6i\theta} + e^{-6i\theta} - 6e^{4i\theta} - 6e^{-4i\theta} + 15e^{2i\theta} + 15e^{-2i\theta} - 20) + \ldots\}$$
$$= \omega^2\{(\tfrac{1}{2}\rho^2 + \tfrac{1}{8}\rho^4 + \tfrac{1}{16}\rho^6 + \tfrac{5}{128}\rho^8 + \ldots)$$
$$+ (\tfrac{1}{4}\rho^2 - \tfrac{1}{32}\rho^6 - \tfrac{1}{64}\rho^8 - \ldots)(e^{2i\theta} + e^{-2i\theta})$$
$$+ (-\tfrac{1}{16}\rho^4 - \tfrac{1}{32}\rho^6 - \tfrac{1}{64}\rho^8 - \ldots)(e^{4i\theta} + e^{-4i\theta})$$
$$+ \left(\frac{\rho^6}{64} + \frac{\rho^8}{64}\right)(e^{6i\theta} + e^{-6i\theta}) + \ldots\}$$
$$= \omega^2(s_o + s_2\cos 2\theta + s_4\cos 4\theta + s_6\cos 6\theta + \ldots)$$

where

$$s_o = \tfrac{1}{2}\rho^2 + \tfrac{1}{8}\rho^4 + \tfrac{1}{16}\rho^6 + \ldots$$
$$s_2 = \tfrac{1}{2}\rho^2 - \tfrac{1}{32}\rho^6 - \tfrac{1}{32}\rho^8 \ldots$$
$$s_4 = -\tfrac{1}{8}\rho^4 - \tfrac{1}{16}\rho^6 - \tfrac{1}{32}\rho^8 \ldots$$
$$s_6 = \tfrac{1}{32}\rho^6 + \tfrac{1}{32}\rho^8 \ldots$$

Example.—Calculate s_o, s_2, s_4, s_6 for the following values of ρ:

$\rho = \frac{1}{3}$. Ans.: ·0571, ·0556, — ·0015, —
$\rho = \frac{1}{4}$. Ans.: ·0317, ·0312, — ·0009, —
$\rho = \frac{1}{5}$. Ans.: ·0202, ·0200, — ·0002, —

19.—Expand $f(\theta) = x^2 = 2rx\cos\theta + l^2 - r^2$

$$\therefore x = r(q_0 + \cos\theta + q_2\cos 2\theta + q_4\cos 4\theta + \ldots) \quad \text{(Sect. 16)}$$

$$\begin{aligned}\therefore x^2 &= l^2 - r^2 + 2r^2(q_0\cos\theta + \cos^2\theta + q_2\cos\theta\cos 2\theta \\ &\qquad + q_4\cos\theta\cos 4\theta + \ldots) \\ &= l^2 - r^2 + 2r^2q_0\cos\theta + r^2\{\cos 2\theta + 1 + q_2(\cos 3\theta + \cos\theta) \\ &\qquad + q_4(\cos 5\theta + \cos 3\theta) + \ldots\} \\ &= l^2 + r^2\{(2q_0 + q_2)\cos\theta + \cos 2\theta + (q_2 + q_4)\cos 3\theta \\ &\qquad + (q_4 + q_6)\cos 5\theta + \ldots\}\end{aligned}$$

By differentiation we obtain

$$\begin{aligned}2x\frac{dx}{dt} = -r^2\omega\{(2q_0 + q_2)\sin\theta + 2\sin 2\theta + 3(q_2 + q_4)\sin 3\theta \\ + 5(q_4 + q_6)\sin 6\theta + \ldots\}.\end{aligned}$$

Note on Foregoing Series.—Referring to Fig. 29, we have

$$a/\sin\phi = c/\sin\psi = c/\sin(\theta + \phi).$$
$$\therefore \sin\phi = \rho\sin(\theta + \phi).$$

(See Examples 8, 10, 11.)

Again,

$$\tan\phi = \frac{a\sin\theta}{c - a\cos\theta} = \frac{\rho\sin\theta}{1 - \rho\cos\theta}.$$

Differentiating,

$$\sec^2\phi\,.\,\frac{d\phi}{dt} = \frac{\rho\cos\theta - \rho^2}{(1 - \rho\cos\theta)^2}\,\frac{d\theta}{dt}.$$

As $\sec.^2\phi = 1 + \tan^2\phi$, this gives

$$\frac{d\phi}{dt} = \frac{\rho\cos\theta - \rho^2}{1 - 2\rho\cos\theta + \rho^2}\,\frac{d\theta}{dt}.$$

(See Examples 2, 9.)

Also, from Fig. 29, we have

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac\cos\theta \\ &= c^2(a^2/c^2 + 1 - 2ac^{-1}\cos\theta).\end{aligned}$$

Hence $$b = c(1 - 2\rho\cos\theta + \rho^2)^{\frac{1}{2}}.$$

(See Examples 13, 14.)

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